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Hausdorff Dimension on Fuzzy Space

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***Abstract:**Fuzzy mathematics forms a branch of mathematics related to fuzzy set theory and fuzzy logic. Hausdorff dimension is a concept in measure theory. It serves as a measure of a local size of a space taking into account the distance between its points. In Hausdorff dimension distance between all members of that set are defined and dimension drawn from the real number R. In this paper we study about the Hausdorff dimension is drawn from fuzzy space. Also we discuss Frostman's lemma on fuzzy space.*

***Keywords:** Fuzzy space, Fuzzy logic, Fuzzy Hausdorff function, Hausdorff dimension, level-n dyadic cubes, Frostman's lemma.*

I. INTRODUCTION

Fuzzy sets can be regarded as a generalization of characteristic functions taking values between 0 and 1(including 0 and 1) and the characteristic function on a set X is the constant mapping taking the whole of X to 1. The point of X posses the same statuses relative to the characteristic function on X, (i.e.), are not distinguishable by characteristic function on X while they are distinguishable by a fuzzy set on X. Hausdorff dimension is a concept in mathematics introduced in 1918 by mathematician Felix Hausdorff. The dimension generalizes the notion of the dimension of a real vector space. That is, the Hausdorff dimension of an n-dimensional inner product space equals n. Hausdorff dimension of a single point is zero, of a line is 1, of a square is 2, and of a cube is 3. For set of points that define a smooth shape or a shape that has a small number of corners, the shapes of traditional geometry and science, the Hausdorff dimension is an integer agreeing with a dimension corresponding to its topology. Usually we find the Hausdorff dimension on the metric space or general space. In this paper we find the Hausdorff dimension on the fuzzy space.

II. PRELIMINARIES

Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive process, such as thinking and reasoning.

A. Some essential characteristics of fuzzy logic:

1) In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
2) Everything is a matter of degree.
3) Knowledge is interpreted a collection of elastic or equivalently, fuzzy constraints on a collection of variables.
4) Inference is viewed as a process of propagation of elastic constraints.
5) Anylogic system can be fuzzified.

There are two main characteristics of fuzzy systems that gives them better performance for specific application:

B. Definition: 1

Let X be a non empty set. A fuzzy set A in X is characterized by its membership function $\mu_A: X \to [0,1]$ and $\mu_A(X)$ is interpreted as the degree of membership of elements x in fuzzy set A for each $x \in X$.

C. Definition: 2

Let A and B be fuzzy set in X, then

$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \forall\ x \in X.$

$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x) \quad \forall\ x \in X.$

$C = A \cup B \Leftrightarrow \mu_A(x) = \max(\mu_A(x), \mu_B(x)) \quad \forall\ x \in X.$

$D = A \cap B \Leftrightarrow \mu_A(x) = \min(\mu_A(x), \mu_B(x)) \quad \forall\ x \in X.\quad E = A' \Leftrightarrow \mu_E(x) = 1 - \mu_A(x), \quad \forall\ x \in X.$

$$A = \{A_i | i \in I\}\ the\ union\ C = \bigcup_I A_i\ and\ intersection\ D = \bigcap_I A_i\ defined\ by$$

$$\mu_C(x) = \sup\{\mu_{A_i}(x)\}\ \forall\ x \in X \text{ and } \mu_D(x) = \inf\{\mu_{A_i}(x)\}\ \forall\ x \in X$$

D. Definition: 3

A fuzzy space X is said to be fuzzy Hausdorff if each $\lambda - diagonal\ d(\lambda)$ of X is a fuzzy closed set of $X \times X$.

E. Definition: 4

For each integer k let Δ_k be the set of cubes in R^n of side length 2^{-k} and corners in the set $2^{-k}Z^n = \{2^{-k}(v_1, v_2, \dots v_n) \ : \ v_j \in Z\}$ and let Δ be the union of all the Δ_k.

F. Definition: 5

Let (X, ρ) be a metric space. For any subset $U \subset X$, let $diam\ U$ denote its diameter, that is $diam\ U = sup\{\rho(x,y)\ |x, y \in U\}$, Let S be any subset of X, and $\delta > 0$ a real number. Define $H_\delta^d(X) = \inf\ (\sum_{i=0}^{\infty}(diam\ U_i)^d : \cup_{i=1}^{\infty} U_i \supseteq S,\ diam\ U_i < \delta)$

G. Definition: 6

The Hausdorff dimension of X is defined by $dim_H(X) = \inf\{d \geq 0 \ : \ H^d(X) = 0\}$

Where $H^d(X) = \lim_{\delta \to 0} H_\delta^d(X)$.

Equivalently $dim_H(X)$ may be defined as the infimum of the set of $d \in [0, \infty)$ such that the d- dimensional Hausdorff measure of x is zer.

H. THEOREM : 6.1

Let X be fuzzy set and A be any subset of X. If $H^p(A) < 1$, then $H^q(A) = 0\ for\ all\ q > p$.

$$H^q(A) = \lim_{\delta \to 0} H_\delta^q(A)$$

$$= \lim_{\delta \to 0} \inf\ (\sum_{\alpha \in I}(diam\ U_\alpha)^q : \cup_{\alpha \in I} U_\alpha \geq A, diam\ U_\alpha < \delta)$$

$$\leq \lim_{\delta \to 0} \inf\ (\sum_{\alpha \in I}(diam\ U_\alpha)^p \delta^{q-p} : \cup_{\alpha \in I} U_\alpha \geq A, diam\ U_\alpha < \delta)$$

$$= \lim_{\delta \to 0} \delta^{q-p} \inf\ (\sum_{\alpha \in I}(diam\ U_\alpha)^p : \cup_{\alpha \in I} U_\alpha \geq A, diam\ U_\alpha < \delta)$$

$$= \lim_{\delta \to 0} \delta^{q-p}\ H^p(A)$$

$H^q(A) = 0$ (let $\delta \to 0$). $for\ all\ q > p$.

Hausdorff dimension of an interval on fuzzy space:

I. THEOREM : 6.2

Let X be the fuzzy space. Let $A = [a, b]$ show that $m(A) = b - a$ which leads to $dim_H(A) = 1$.

*1) Proof***:**Let X be any fuzzy space and $A = [a, b]$

First we prove $m(A) \leq b - a$.

Let $\in > 0$ and N bean integer such that $h = \frac{b-a}{N} < \frac{\in}{2}$.

Let $x_i = a + ih$ where $i = 1,2,3 \dots N$.

We define $C = \{C_i,\ \ i = 1,2,3 \dots N\}$ where C is an open cover for A.

Assume ɳ $< b - a$ $\quad C_1 = \left[a, x_1 + \frac{ɳ}{N}\right]$

$$C_i = \left[x_i - \frac{ɳ}{2N}, x_{i+1} + \frac{ɳ}{2N}\right]$$

$$C_i = \left[x_{N-1} - \frac{ɳ}{N},\ b\right]$$

For each i, $\quad diam\ (C_i) = h + \frac{ɳ}{N} < \frac{\in}{2} + \frac{\in}{2} = \in$.

Thus $m(C) < \in$ and $\sum diam\ (C_i) = h + \frac{ɳ}{N} + (N-2)\left(h + \frac{ɳ}{N}\right) + \left(h + \frac{ɳ}{N}\right)$

$$= hN + ɳ$$

$$= b - a + ɳ$$

$$m(A) \leq b - a + ɳ \quad \text{for every } ɳ > 0$$

$$\Rightarrow \quad m(A) \leq b - a.$$

Now show that $m(A) \geq b - a$.

Let $C = \{C_j \ : \ j \in Z_+\}$ be an open cover of Aɳ

Since A is a compact and $\in > 0$

$$|x-y| < \in \quad for\ every\ \ x,y \in A$$

Then there exist $C_i \in C$ such that $x, y \in C_i$.

Let N be a positive integer $h = \frac{b-a}{N} < \in$.

Define $x_i = a + ih\ for\ i = 1,2 \ldots N$

$x_i - x_{i-1} = h \leq diam\ (C_j(i))$

Hence $b - a = \sum_{i=1}^{N} x_i - x_{i-1} < \sum diam\ \left(C_j(i)\right)$

$b - a \leq diam\ (C_j(i))$

$b - a \leq m(A)$

Thus $m(A) \geq b - a$.

Hence $m(A) = b - a$.

J. *Elementary Properties of Hausdorff Dimension on Fuzzy Space*

It has the following properties

1) Let X and Y be any two fuzzy space, If $X \subset Y$ then $dim_H(X) \leq dim_H(Y)$
2) If X_i is a countable collection of fuzzy sets with $dim_H(X_i) \leq d$ then $dim_H(\cup_i X_i) \leq d$.
3) If a fuzzy space X is countable then $dim_H(X) = 0$.
4) Let X be a fuzzy space, if $dim_H(X) = d$ and $dim_H(Y) = d'$,

$$then\ dim_H(X \times Y) \geq d + d'.$$

Let X be a fuzzy space, if $f: X \to f(x)$ is a lipschitz map,

then $dim_H\left(f(x)\right) \leq dim_H(X)$.

K. *Theorem: 6.3 (Frostman's Lemma)*

Let X be a fuzzy space. If $X \subseteq \mathbb{R}^d$ is closed and $H_\infty^\propto(X) > 0$, then there is a regular probability measure supported on X.

1) *Proof:*

Let X be a fuzzy space and $X \subseteq \mathbb{R}^d$ is closed.

Withoutloss of generality we assume that $X \subseteq [0,1]^d$.

Consider X as intersection with each of the level-0dyadic cubes.

$$X = \bigcup_{D \in D_0} X \cap \overline{D}$$

By Hausdorff Property

If $H_\infty^\propto(X \cap \overline{D}) = 0$, for each D in the union then $H_\infty^\propto = 0$.

Thus $H_\infty^\propto(X \cap \overline{D}) > 0$ for $D \in D_0$

By translating X we may assume that $\overline{D} = [0,1]^d$.

Let $A = [0,1]^d$ and let $\Pi: A^N \to [0,1]^d$

Denote the map $\Pi(\omega) = \sum_{i=1}^{\infty} 2^{-n}\ \omega_n$.

For d=1

$\Pi: [\{0,1\}]^N \to [0,1]$

This map Π_0 gives the binary representation.

For d >1, Π maps each components of $\{0,1\}^N\ in\ (\{0,1\}^N)^d$.

Thus $(\{0,1\}^d)^N \cong (\{0,1\}^N)^d$

Therefore the map Π is onto, since each co-ordinates x_i in binary representation as

$x_i = 0.x_{1n}x_{2n}\,x_{3n} \ldots\ x_{dn}$ and set $\omega_n = \{x_{1n}, x_{2n}, x_{3n}, \ldots, x_{dn}\}$

But Π is not one-to-one if x has co-ordinates which are dyadic rational. Because there will be multiple pre-images.

The space A^N can be given the metric

$d(\omega, \eta) = 2^{-n}$ for $n = min\{k \geq 0\ :\ \omega_{k+1} \neq \eta_{k+1}\}$

By lipschitz map Π is compact and continuous. Thus every closed subset $X \subseteq [0,1]^d$ transforms into a closed subset $\Pi^{-1}(X)$ of A^N.

Conversely every closed subset of $Y \subseteq A^N$ represents as $X = \Pi(Y)$ it is closed subset of $[0,1]^d$.

For $\omega_1, \omega_2, \dots, \omega_n \in A$, the cylinder set $[\omega_1, \omega_2, \dots, \omega_n] \subseteq A^N$ is
$[\omega_1, \omega_2, \dots, \omega_n] = \eta \in A^N : \eta_1, \eta_2, \dots, \eta_n = \omega_1, \omega_2, \dots, \omega_n$
Choose arbitrary $\mathcal{E}$, thus $[\mathcal{E}] = A^N$.
The metric d is defined, so that $[\omega_1, \omega_2, \dots, \omega_n] = B_{2^{-n}}(\eta)$ for every $\eta = [\omega_1, \omega_2, \dots, \omega_n]$ and the diameter of this ball is 2^{-n}. so cylinder set is closed.
For each family of sets $C_n = \{[a] : a \in A^N\}$ forms a finite partition of A^N.
Complement of $[a]$ is the union of finitely many closed sets, so cylinder sets are also open.
The image of $\Pi[\omega_1, \omega_2, \dots, \omega_n]$ is the closure of the dyadic cube $D \in D_0$ containing $\sum_{i=1}^{n} 2^{-i}\omega_i = 0.\omega_1.\omega_2 \dots \omega_n$ which is a set of diameter $\sqrt{d}.2^{-n}$. the pre-image $\Pi^{-1}(D)$ of any level-n dyadic cell $D \in D_n$ intersects at most 2^d level-n cylinder sets.

III. CONCLUSION

Fuzzification of mathematical concepts is based on a generalization of these concepts from characteristic functions to membership functions. Fuzzification is the process of changing a real scalar values into a fuzzy values. In fuzzy space the Hausdorff dimension value is normalized to unity. Fuzzification gives more clarity on finding dimension.

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