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ϵ -Best Approximation and E- Orthogonality

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Abstract: The purpose of this paper is to study the concept of ϵ -Best approximation and ϵ -orthogonality. I discussed their properties and noted that are similar to the properties of best approximation.

Keywords: ϵ -Best approximation, normed linear spaces, proximal, ϵ -orthogonality, convex.

I. INTRODUCTION

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications

What do we mean by "**Best approximation**" in normed linear spaces?

To explain this, let X be a normed linear space, and let G be a nonempty subset of X . An element $g_0 \in G$ is called a best approximation to x from G if g_0 is closest to x from among all the elements of G .

That is, $\|x - g_0\| \leq \|x - g\|$ for all $g \in G$.

The set of all such elements $g_0 \in G$ are called a **best approximation** to $x \in X$ is denoted by $P_G(x)$.

If $P_G(x)$ contains at least one element, then the subset G is called a **proximal** set. If

each element $x \in X$ has a unique best approximation in G , then G is called a **Chebyshev** set of X .

The theory of approximation is mainly concerned with the following fundamental questions.

- 1) (*Existence of best approximation*) Which subsets are proximal?
- 2) (*Uniqueness of best approximation*) Which subsets are Chebyshev?
- 3) (*Characterization of best approximation*) How to recognize when a given $y \in G$ is a best approximation to x or not?
- 4) (*Error of approximation*) How to compute the error of approximation $d(x, G)$?
- 5) (*Computation of best approximation*) How to describe some useful algorithms for actually computing best approximation?
- 6) (*Continuity of best approximation*) How does the set of all best approximation vary as a function of x or (G) ?

A. Definition 1.1[1]

Let G be a nonempty subset of a real normed linear space E and let an element $f \in E$ be given. The problem of **best approximation** is to determine an element $g_f \in G$ such that

$$\|f - g_f\| = \inf_{g \in G} \|f - g\|$$

such an element is called a **best approximation** to f from G , and

$$d(f, G) = \inf_{g \in G} \|f - g\| \text{ is called the } \textit{minimal deviation} \text{ off from } G.$$

The set of all elements $g_0 \in G$ that are called best approximation to $x \in X$ is

$$P_G(x) = \{ g_0 \in G: \|x - g_0\| \leq \|x - g\| \text{ for all } g \in G \}$$

Hence P_G defines a mapping from X into the power set of G is called the **metric projection** onto G , (other names nearest point mapping, proximity map)

B. Remark 1.2[1]

The set $P_G(x)$ of all **best approximation** to $x \in X$ can be written as

$$P_G(x) = \{ g_0 \in G: \|x - g_0\| = d(x, G) \}$$

C. *Definition 1.3.[3]*

A set S , in a linear space is **convex** .if $s_1, s_2 \in S$ implies that

$$\lambda_1 s_1 + \lambda_2 s_2 \in S$$

If λ_1 and λ_2 are non negative and $\lambda_1 + \lambda_2 = 1$

If S is empty or consists of one point, then it is clearly *convex*

D. *Definition 1.4[1]*

If $P_G(x)$ contains at least one element, then the subset G is called a *proximal set*.

In other words, if $P_G(x) \neq \emptyset$ then G is called a *proximal set*

The term *proximal set* (is a combination of proximity and maximal)

E. *Definition 1.5[1] (Quasi-Orthogonal Set)*

Let X be a normed linear space, and G a nonempty subset of X . Then we say that G is *quasi-orthogonal set* if $G \perp_B \hat{G}$, that is $g \perp_B \hat{G}$ for every $g \in G$.

$$\text{where } \hat{G} = \{x \in X; \|x\| = d(x, G)\} = \{x \in X : x \perp_B G\}.$$

F. *Remark 1.6[1]*

In a Hilbert space, any closed subspace is quasi-orthogonal.

Proof:

Let H be a Hilbert space and G a closed subspace of H .

Then $\hat{G} = G^\perp = \{y \in H : \langle x, y \rangle = 0, \text{ for all } x \in G\}$. Then $G \perp \hat{G}$.

Therefore G is quasi-orthogonal subspace of H .

G. *Definition 1.7[2]*

Let X be a normed linear space and G be a subset of X , and $\varepsilon > 0$. A point $g_0 \in G$ is said to be ε -best approximation for $x \in X$ if and only if

$$\|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G$$

H. *Remark 1.8[2]*

For $x \in X$, the set of all ε -Best approximation of x in G is denoted by

$P_G(x, \varepsilon)$, in other words,

$$P_G(x, \varepsilon) = \{g_0 \in G : \|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G\}.$$

I. *Theorem 1.9[2]*

Let G be a subspace of a normed linear space X . Then $P_G(x, \varepsilon)$ is bounded.

Proof:

Let $g_1, g_2 \in P_G(x, \varepsilon)$, then $\|x - g_1\| \leq \|x - g\| + \varepsilon$ for all $g \in G$, and

$$\|x - g_2\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G$$

$$\text{Now, } \|g_1 - g_2\| = \|g_1 - x + x - g_2\| \leq \|x - g_1\| + \|x - g_2\|$$

$$\leq \|x - g\| + \varepsilon + \|x - g\| + \varepsilon = 2\|x - g\| + 2\varepsilon = k,$$

so we have $\|g_1 - g_2\| \leq k$ where $k = 2d(x, G) + 2\varepsilon$.

Therefore, $P_G(x, \varepsilon)$ is bounded.

Hence the proof

J. Theorem 1.10[2]

Let G be a subspace of normed linear space X , and $x \in X$. Then $P_G(x, \epsilon)$ is convex.

Proof:

Let $g_1, g_2 \in P_G(x, \epsilon)$, and $0 \leq \lambda \leq 1$, then $\|x - g_1\| \leq \|x - g\| + \epsilon$ for all $g \in G$, and

$$\|x - g_2\| \leq \|x - g\| + \epsilon \text{ for all } g \in G$$

$$\begin{aligned} \text{Now, } \|x - (\lambda g_1 + (1 - \lambda) g_2)\| &= \|x - \lambda g_1 - g_2 + \lambda g_2\| \\ &= \|x - \lambda g_1 - g_2 + \lambda g_2 + \lambda x - \lambda x\| \\ &= \|\lambda(x - g_1) + (1 - \lambda)(x - g_2)\| \\ &\leq \lambda \|x - g_1\| + (1 - \lambda) \|x - g_2\| \\ &\leq \lambda(\|x - g\| + \epsilon) + (1 - \lambda)(\|x - g\| + \epsilon) \\ &= \|x - g\| + \epsilon. \end{aligned}$$

Thus, $\lambda g_1 + (1 - \lambda) g_2 \in P_G(x, \epsilon)$.

Hence $P_G(x, \epsilon)$ is convex.

Hence the proof

K. Definition 1.11.[2] (ϵ -orthogonality)

Let X be a normed linear space, $\epsilon > 0$, and $x, y \in X$. We call x is ϵ -orthogonal to y and is denoted by $x \perp_\epsilon y$ if and only if

$$\|x + \alpha y\| + \epsilon \geq \|x\| \text{ for all scalar } \alpha \text{ with } |\alpha| \leq 1$$

For subsets G_1, G_2 of X , $G_1 \perp_\epsilon G_2$ if and only if, $g_1 \perp_\epsilon g_2$ for all $g_1 \in G_1, g_2 \in G_2$.

L. Theorem: 1.12[2]

Let X be a normed linear space, G be a subspace of X , and $\epsilon > 0$. Then for all $x \in X$, $g_0 \in P_G(x, \epsilon)$ if and only if $(x - g_0) \perp_\epsilon G$.

Proof:

(\Rightarrow) Suppose $g_0 \in P_G(x, \epsilon)$. Put $g_1 = g_0 - \alpha g$ for $g \in G$ and $|\alpha| \leq 1$.

Since $g_0 \in P_G(x, \epsilon)$ and $g_1 \in G$ so, then, $\|x - g_0\| \leq \|x - g_1\| + \epsilon$, then

$$\|x - g_0\| \leq \|x - (g_0 - \alpha g)\| + \epsilon, \text{ and this implies that}$$

$$\|x - g_0\| \leq \|(x - g_0) + \alpha g\| + \epsilon.$$

Therefore, $(x - g_0) \perp_\epsilon G$.

(\Leftarrow) Let $(x - g_0) \perp_\epsilon G$, then for all α with $|\alpha| \leq 1$ and $g_1 \in G$ we have,

$$\|x - g_0\| \leq \|x - g_0 + \alpha g_1\| + \epsilon$$

For any $g \in G$ by putting $g_1 = g_0 - g$ and $\alpha = 1$, the last inequality implies,

$$\|x - g_0\| \leq \|x - g\| + \epsilon$$

Therefore, $g_0 \in P_G(x, \epsilon)$

Hence the proof

M. Notation 1.13

Let X be a normed linear space, and G a subspace of X , and for $\epsilon > 0$, let

$$P_G^{-1}(0, \epsilon) = \{x \in X: \|x\| \leq \|x - g\| + \epsilon \text{ for all } g \in G\} = \{x \in X: x \perp_\epsilon G\}$$

Then, $\hat{G}_\epsilon = \{x \in X: x \perp_\epsilon G\}$.

N. Lemma 1.14[2]

Let G be a subspace of a normed linear space X . Then for all $x \in X$ and all $\epsilon > 0$, we have, $g_0 \in P_G(x, \epsilon)$ if and only if $(x - g_0) \in \hat{G}_\epsilon$

Proof:

$g_0 \in P_G(x, \epsilon)$ if and only if by [Theorem 1.12], $(x - g_0) \perp_\epsilon \hat{G}_\epsilon$ if and only if $(x - g_0) \in \hat{G}_\epsilon$.

O. Corollary 1.15

Let G be a subspace of a normed linear space X , and let $\epsilon > 0, x \in X$. Then,

$$P_G(x, \epsilon) = G \cap (x - \hat{G}_\epsilon)$$

Proof:

$g_0 \in G \cap (x - \hat{G}_\varepsilon)$ if and only if $g_0 \in G$, and $g_0 \in (x - \hat{G}_\varepsilon)$ if and only if $g_0 \in G$ and $g_0 = x - \hat{g}$, where $\hat{g} \in \hat{G}_\varepsilon$ if and only if $g_0 \in G$, $\hat{g} = (x - g_0) \in \hat{G}_\varepsilon$ if and only if $g_0 \in P_G(x, \varepsilon)$ by [Lemma 1.14].

Therefore, $P_G(x, \varepsilon) = G \cap (x - \hat{G}_\varepsilon)$

Hence the proof

P. Theorem 1.16

Let G be a subspace of a normed linear space X , $\varepsilon > 0$, and $\varepsilon \geq \alpha$. Then,

$$\hat{G} \subset \hat{G}_\alpha \subset \hat{G}_\varepsilon, \text{ and therefore } \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$$

Proof:

Let $x \in \hat{G}$, then $\|x\| \leq \|x - g\|$ for all $g \in G$.

Now $\|x\| \leq \|x - g\| \leq \|x - g\| + \alpha$ [$\alpha > 0$], so, we have $x \in \hat{G}_\alpha$.

Hence $\hat{G} \subset \hat{G}_\alpha$ (1)

Let $x \in \hat{G}_\alpha$, then $\|x\| \leq \|x - g\| + \alpha \leq \|x - g\| + \varepsilon$ [$\varepsilon > \alpha$], this implies that $x \in \hat{G}_\varepsilon$, and so, $\hat{G}_\alpha \subset \hat{G}_\varepsilon$ (2)

(1) and (2) together imply that $\hat{G} \subset \hat{G}_\alpha \subset \hat{G}_\varepsilon$,

Now, we show $\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$

From above we have $\hat{G} \subset \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon$

conversely, let $x \in \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon$,

Then for all $\varepsilon > 0$, $0 \leq \|x\| \leq \|x - g\| + \varepsilon$ for all $g \in G$, then for all $n \in \mathbb{N}$,

$$0 \leq \|x\| \leq \|x - g\| + \frac{1}{n} \text{ for all } g \in G:$$

As $n \rightarrow \infty$, $\|x\| \leq \|x - g\|$ for all $g \in G$, then $x \in \hat{G}$,

and so,

$$\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon \subset \hat{G}$$

Therefore $\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$

Hence the proof.

Q. Lemma 1.17

Let G be a subspace of a normed linear space X . Then.

- 1) If $\varepsilon > 0$, $x, g \in X$ and $x \perp_\varepsilon g$, then $x \perp_\delta g$ for all $\delta \geq \varepsilon$.
- 2) If $x, g \in X$ and $x \perp_B g$, then $x \perp_\varepsilon g$ for all $\varepsilon > 0$.
- 3) If $x \in X$, and $\varepsilon > 0$, then $0 \perp_\varepsilon x$, $x \perp_\varepsilon 0$.
- 4) If $x \perp_\varepsilon g$ and $|\beta| < 1$, then $\beta x \perp_\varepsilon \beta g$.

Proof:

(a) Let $\varepsilon > 0$, $x, g \in X$ and $x \perp_\varepsilon g$, then by [Definition 1.11] we have

$$\|x\| \leq \|x + \alpha g\| + \varepsilon, \text{ where } |\alpha| \leq 1 \text{ and } \varepsilon > 0$$

Then, $\|x\| \leq \|x + \alpha g\| + \varepsilon \leq \|x + \alpha g\| + \delta$, [since $\delta \geq \varepsilon$]

Therefore, $x \perp_\delta g$

(b) Let $x, g \in X$ and $x \perp_B g$, then $\|x\| \leq \|x + \alpha g\|$ for all $\alpha \in \mathbb{R}$

Since $\varepsilon > 0$, then $\|x\| \leq \|x + \alpha g\| \leq \|x + \alpha g\| + \varepsilon$ for all $|\alpha| \leq 1$

Hence $x \perp_\varepsilon g$ for all $\varepsilon > 0$

(c) Let $x \in X$ and $\varepsilon > 0$, then $\|0\| \leq \|0 + \alpha x\| + \varepsilon$, and so $0 \perp_\varepsilon x$.

We have also $\|x\| \leq \|x\| + \varepsilon$, then $\|x\| \leq \|x + \alpha 0\| + \varepsilon$,

Hence $x \perp_\varepsilon 0$.

(d) Let $x \perp_\varepsilon g$, and $|\beta| < 1$, then $\|x\| \leq \|x + \alpha g\| + \varepsilon$.

Multiply both sides by $|\beta|$,

$$\text{we get } |\beta| \|x\| \leq |\beta| \|x + \alpha g\| + |\beta| \varepsilon$$

$$\leq |\beta| \|x + \alpha g\| + \varepsilon, \text{ and so}$$

$$\|\beta x\| \leq \|\beta x + \alpha \beta g\| + \varepsilon$$

Therefore, $\beta x \perp_\varepsilon \beta g$

Hence the proof

R. Theorem 1.18

Let G be a subspace of a normed linear space X . If $x \in X$, $\varepsilon > 0$ and $\delta \geq \varepsilon$, then $P_G(x, \varepsilon) \subset P_G(x, \delta)$.

Proof:

Let $g_0 \in P_G(x, \varepsilon)$. Then by [Definition 1.7], we have

$$\|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G \text{ and } \varepsilon > 0$$

$$\text{Then } \|x - g_0\| \leq \|x - g\| + \varepsilon \leq \|x - g\| + \delta$$

[since $\delta > \varepsilon$], then, $g_0 \in P_G(x, \delta)$.

Therefore $P_G(x, \varepsilon) \subset P_G(x, \delta)$

Hence the proof

II. CONCLUSION

Here, I conclude my paper as ε -Best approximation and ε -orthogonality has the properties which are similar to the properties of best approximation.

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