



# IJRASET

International Journal For Research in  
Applied Science and Engineering Technology



---

# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

---

**Volume:** TPAM-2018 **Issue:** conference **Month of publication:** March 2018

**DOI:**

[www.ijraset.com](http://www.ijraset.com)

Call:  08813907089

E-mail ID: [ijraset@gmail.com](mailto:ijraset@gmail.com)

# $\epsilon$ -Best Approximation and E- Orthogonality

R.S. Karunya

Assistant Professor, Department of Mathematics, St. Joseph's College of Arts and Science for Women, Hosur.

**Abstract:** The purpose of this paper is to study the concept of  $\epsilon$ -Best approximation and  $\epsilon$ -orthogonality. I discussed their properties and noted that are similar to the properties of best approximation.

**Keywords:**  $\epsilon$ -Best approximation, normed linear spaces, proximal,  $\epsilon$ -orthogonality, convex.

## I. INTRODUCTION

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications

What do we mean by "**Best approximation**" in normed linear spaces?

To explain this, let  $X$  be a normed linear space, and let  $G$  be a nonempty subset of  $X$ . An element  $g_0 \in G$  is called a best approximation to  $x$  from  $G$  if  $g_0$  is closest to  $x$  from among all the elements of  $G$ .

That is,  $\|x - g_0\| \leq \|x - g\|$  for all  $g \in G$ .

The set of all such elements  $g_0 \in G$  are called a **best approximation** to  $x \in X$  is denoted by  $P_G(x)$ .

If  $P_G(x)$  contains at least one element, then the subset  $G$  is called a **proximal** set. If

each element  $x \in X$  has a unique best approximation in  $G$ , then  $G$  is called a **Chebyshev** set of  $X$ .

The theory of approximation is mainly concerned with the following fundamental questions.

- 1) (*Existence of best approximation*) Which subsets are proximal?
- 2) (*Uniqueness of best approximation*) Which subsets are Chebyshev?
- 3) (*Characterization of best approximation*) How to recognize when a given  $y \in G$  is a best approximation to  $x$  or not?
- 4) (*Error of approximation*) How to compute the error of approximation  $d(x, G)$ ?
- 5) (*Computation of best approximation*) How to describe some useful algorithms for actually computing best approximation?
- 6) (*Continuity of best approximation*) How does the set of all best approximation vary as a function of  $x$  or  $(G)$ ?

### A. Definition 1.1[1]

Let  $G$  be a nonempty subset of a real normed linear space  $E$  and let an element  $f \in E$  be given. The problem of **best approximation** is to determine an element  $g_f \in G$  such that

$$\|f - g_f\| = \inf_{g \in G} \|f - g\|$$

such an element is called a **best approximation** to  $f$  from  $G$ , and

$$d(f, G) = \inf_{g \in G} \|f - g\| \text{ is called the } \textit{minimal deviation} \text{ off from } G.$$

The set of all elements  $g_0 \in G$  that are called best approximation to  $x \in X$  is

$$P_G(x) = \{ g_0 \in G: \|x - g_0\| \leq \|x - g\| \text{ for all } g \in G \}$$

Hence  $P_G$  defines a mapping from  $X$  into the power set of  $G$  is called the **metric projection** onto  $G$ , (other names nearest point mapping, proximity map)

### B. Remark 1.2[1]

The set  $P_G(x)$  of all **best approximation** to  $x \in X$  can be written as

$$P_G(x) = \{ g_0 \in G: \|x - g_0\| = d(x, G) \}$$

C. *Definition 1.3.[3]*

A set  $S$ , in a linear space is **convex** .if  $s_1, s_2 \in S$  implies that

$$\lambda_1 s_1 + \lambda_2 s_2 \in S$$

If  $\lambda_1$  and  $\lambda_2$  are non negative and  $\lambda_1 + \lambda_2 = 1$

If  $S$  is empty or consists of one point, then it is clearly *convex*

D. *Definition 1.4[1]*

If  $P_G(x)$  contains at least one element, then the subset  $G$  is called a *proximal set*.

In other words, if  $P_G(x) \neq \emptyset$  then  $G$  is called a *proximal set*

The term *proximal set* (is a combination of proximity and maximal)

E. *Definition 1.5[1] (Quasi-Orthogonal Set)*

Let  $X$  be a normed linear space, and  $G$  a nonempty subset of  $X$ . Then we say that  $G$  is *quasi-orthogonal set* if  $G \perp_B \hat{G}$ , that is  $g \perp_B \hat{G}$  for every  $g \in G$ .

$$\text{where } \hat{G} = \{x \in X; \|x\| = d(x, G)\} = \{x \in X : x \perp_B G\}.$$

F. *Remark 1.6[1]*

In a Hilbert space, any closed subspace is quasi-orthogonal.

*Proof:*

Let  $H$  be a Hilbert space and  $G$  a closed subspace of  $H$ .

Then  $\hat{G} = G^\perp = \{y \in H : \langle x, y \rangle = 0, \text{ for all } x \in G\}$ . Then  $G \perp \hat{G}$ .

Therefore  $G$  is quasi-orthogonal subspace of  $H$ .

G. *Definition 1.7[2]*

Let  $X$  be a normed linear space and  $G$  be a subset of  $X$ , and  $\varepsilon > 0$ . A point  $g_0 \in G$  is said to be  $\varepsilon$ -best approximation for  $x \in X$  if and only if

$$\|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G$$

H. *Remark 1.8[2]*

For  $x \in X$ , the set of all  $\varepsilon$ -Best approximation of  $x$  in  $G$  is denoted by

$P_G(x, \varepsilon)$ , in other words,

$$P_G(x, \varepsilon) = \{g_0 \in G : \|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G\}.$$

I. *Theorem 1.9[2]*

Let  $G$  be a subspace of a normed linear space  $X$ . Then  $P_G(x, \varepsilon)$  is bounded.

*Proof:*

Let  $g_1, g_2 \in P_G(x, \varepsilon)$ , then  $\|x - g_1\| \leq \|x - g\| + \varepsilon$  for all  $g \in G$ , and

$$\|x - g_2\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G$$

$$\text{Now, } \|g_1 - g_2\| = \|g_1 - x + x - g_2\| \leq \|x - g_1\| + \|x - g_2\|$$

$$\leq \|x - g\| + \varepsilon + \|x - g\| + \varepsilon = 2\|x - g\| + 2\varepsilon = k,$$

so we have  $\|g_1 - g_2\| \leq k$  where  $k = 2d(x, G) + 2\varepsilon$ .

Therefore,  $P_G(x, \varepsilon)$  is bounded.

Hence the proof

*J. Theorem 1.10[2]*

Let  $G$  be a subspace of normed linear space  $X$ , and  $x \in X$ . Then  $P_G(x, \epsilon)$  is convex.

*Proof:*

Let  $g_1, g_2 \in P_G(x, \epsilon)$ , and  $0 \leq \lambda \leq 1$ , then  $\|x - g_1\| \leq \|x - g\| + \epsilon$  for all  $g \in G$ , and

$$\|x - g_2\| \leq \|x - g\| + \epsilon \text{ for all } g \in G$$

$$\begin{aligned} \text{Now, } \|x - (\lambda g_1 + (1 - \lambda) g_2)\| &= \|x - \lambda g_1 - g_2 + \lambda g_2\| \\ &= \|x - \lambda g_1 - g_2 + \lambda g_2 + \lambda x - \lambda x\| \\ &= \|\lambda(x - g_1) + (1 - \lambda)(x - g_2)\| \\ &\leq \lambda \|x - g_1\| + (1 - \lambda) \|x - g_2\| \\ &\leq \lambda(\|x - g\| + \epsilon) + (1 - \lambda)(\|x - g\| + \epsilon) \\ &= \|x - g\| + \epsilon. \end{aligned}$$

Thus,  $\lambda g_1 + (1 - \lambda) g_2 \in P_G(x, \epsilon)$ .

Hence  $P_G(x, \epsilon)$  is convex.

Hence the proof

*K. Definition 1.11.[2] ( $\epsilon$ -orthogonality)*

Let  $X$  be a normed linear space,  $\epsilon > 0$ , and  $x, y \in X$ . We call  $x$  is  $\epsilon$ -orthogonal to  $y$  and is denoted by  $x \perp_\epsilon y$  if and only if

$$\|x + \alpha y\| + \epsilon \geq \|x\| \text{ for all scalar } \alpha \text{ with } |\alpha| \leq 1$$

For subsets  $G_1, G_2$  of  $X$ ,  $G_1 \perp_\epsilon G_2$  if and only if,  $g_1 \perp_\epsilon g_2$  for all  $g_1 \in G_1, g_2 \in G_2$ .

*L. Theorem: 1.12[2]*

Let  $X$  be a normed linear space,  $G$  be a subspace of  $X$ , and  $\epsilon > 0$ . Then for all  $x \in X$ ,  $g_0 \in P_G(x, \epsilon)$  if and only if  $(x - g_0) \perp_\epsilon G$ .

*Proof:*

( $\Rightarrow$ ) Suppose  $g_0 \in P_G(x, \epsilon)$ . Put  $g_1 = g_0 - \alpha g$  for  $g \in G$  and  $|\alpha| \leq 1$ .

Since  $g_0 \in P_G(x, \epsilon)$  and  $g_1 \in G$  so, then,  $\|x - g_0\| \leq \|x - g_1\| + \epsilon$ , then

$$\|x - g_0\| \leq \|x - (g_0 - \alpha g)\| + \epsilon, \text{ and this implies that}$$

$$\|x - g_0\| \leq \|(x - g_0) + \alpha g\| + \epsilon.$$

Therefore,  $(x - g_0) \perp_\epsilon G$ .

( $\Leftarrow$ ) Let  $(x - g_0) \perp_\epsilon G$ , then for all  $\alpha$  with  $|\alpha| \leq 1$  and  $g_1 \in G$

we have,

$$\|x - g_0\| \leq \|x - g_0 + \alpha g_1\| + \epsilon$$

For any  $g \in G$  by putting  $g_1 = g_0 - g$  and  $\alpha = 1$ , the last inequality implies,

$$\|x - g_0\| \leq \|x - g\| + \epsilon$$

Therefore,  $g_0 \in P_G(x, \epsilon)$

Hence the proof

*M. Notation 1.13*

Let  $X$  be a normed linear space, and  $G$  a subspace of  $X$ , and for  $\epsilon > 0$ , let

$$P_G^{-1}(0, \epsilon) = \{x \in X : \|x\| \leq \|x - g\| + \epsilon \text{ for all } g \in G\} = \{x \in X : x \perp_\epsilon G\}$$

Then,  $\hat{G}_\epsilon = \{x \in X : x \perp_\epsilon G\}$ .

*N. Lemma 1.14[2]*

Let  $G$  be a subspace of a normed linear space  $X$ . Then for all  $x \in X$  and all  $\epsilon > 0$ , we have,  $g_0 \in P_G(x, \epsilon)$  if and only if  $(x - g_0) \in \hat{G}_\epsilon$

**Proof:**

$g_0 \in P_G(x, \epsilon)$  if and only if by [Theorem 1.12],  $(x - g_0) \perp_\epsilon G$  if and only if  $(x - g_0) \in \hat{G}_\epsilon$ .

*O. Corollary 1.15*

Let  $G$  be a subspace of a normed linear space  $X$ , and let  $\epsilon > 0, x \in X$ . Then,

$$P_G(x, \epsilon) = G \cap (x - \hat{G}_\epsilon)$$

*Proof:*

$g_0 \in G \cap (x - \hat{G}_\varepsilon)$  if and only if  $g_0 \in G$ , and  $g_0 \in (x - \hat{G}_\varepsilon)$  if and only if  $g_0 \in G$  and  $g_0 = x - \hat{g}$ , where  $\hat{g} \in \hat{G}_\varepsilon$  if and only if  $g_0 \in G$ ,  $\hat{g} = (x - g_0) \in \hat{G}_\varepsilon$  if and only if  $g_0 \in P_G(x, \varepsilon)$  by [ Lemma 1.14].

Therefore,  $P_G(x, \varepsilon) = G \cap (x - \hat{G}_\varepsilon)$

Hence the proof

*P. Theorem 1.16*

Let  $G$  be a subspace of a normed linear space  $X$ ,  $\varepsilon > 0$ , and  $\varepsilon \geq \alpha$ . Then,

$$\hat{G} \subset \hat{G}_\alpha \subset \hat{G}_\varepsilon, \text{ and therefore } \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$$

*Proof:*

Let  $x \in \hat{G}$ , then  $\|x\| \leq \|x - g\|$  for all  $g \in G$ .

Now  $\|x\| \leq \|x - g\| \leq \|x - g\| + \alpha$  [ $\alpha > 0$ ], so, we have  $x \in \hat{G}_\alpha$ .

Hence  $\hat{G} \subset \hat{G}_\alpha$  .....(1)

Let  $x \in \hat{G}_\alpha$ , then  $\|x\| \leq \|x - g\| + \alpha \leq \|x - g\| + \varepsilon$  [ $\varepsilon > \alpha$ ], this implies that  $x \in \hat{G}_\varepsilon$ , and so,  $\hat{G}_\alpha \subset \hat{G}_\varepsilon$  .....(2)

(1) and (2) together imply that  $\hat{G} \subset \hat{G}_\alpha \subset \hat{G}_\varepsilon$ ,

Now, we show  $\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$

From above we have  $\hat{G} \subset \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon$

conversely, let  $x \in \bigcap_{\varepsilon > 0} \hat{G}_\varepsilon$ ,

Then for all  $\varepsilon > 0$ ,  $0 \leq \|x\| \leq \|x - g\| + \varepsilon$  for all  $g \in G$ , then for all  $n \in \mathbb{N}$ ,

$$0 \leq \|x\| \leq \|x - g\| + \frac{1}{n} \text{ for all } g \in G:$$

As  $n \rightarrow \infty$ ,  $\|x\| \leq \|x - g\|$  for all  $g \in G$ , then  $x \in \hat{G}$ ,

and so,

$$\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon \subset \hat{G}$$

Therefore  $\bigcap_{\varepsilon > 0} \hat{G}_\varepsilon = \hat{G}$

Hence the proof.

*Q. Lemma 1.17*

Let  $G$  be a subspace of a normed linear space  $X$ . Then.

- 1) If  $\varepsilon > 0$ ,  $x, g \in X$  and  $x \perp_\varepsilon g$ , then  $x \perp_\delta g$  for all  $\delta \geq \varepsilon$ .
- 2) If  $x, g \in X$  and  $x \perp_B g$ , then  $x \perp_\varepsilon g$  for all  $\varepsilon > 0$ .
- 3) If  $x \in X$ , and  $\varepsilon > 0$ , then  $0 \perp_\varepsilon x$ ,  $x \perp_\varepsilon 0$ .
- 4) If  $x \perp_\varepsilon g$  and  $|\beta| < 1$ , then  $\beta x \perp_\varepsilon \beta g$ .

*Proof:*

(a) Let  $\varepsilon > 0$ ,  $x, g \in X$  and  $x \perp_\varepsilon g$ , then by [Definition 1.11] we have

$$\|x\| \leq \|x + \alpha g\| + \varepsilon, \text{ where } |\alpha| \leq 1 \text{ and } \varepsilon > 0$$

Then,  $\|x\| \leq \|x + \alpha g\| + \varepsilon \leq \|x + \alpha g\| + \delta$ , [since  $\delta \geq \varepsilon$ ]

Therefore,  $x \perp_\delta g$

(b) Let  $x, g \in X$  and  $x \perp_B g$ , then  $\|x\| \leq \|x + \alpha g\|$  for all  $\alpha \in \mathbb{R}$

Since  $\varepsilon > 0$ , then  $\|x\| \leq \|x + \alpha g\| \leq \|x + \alpha g\| + \varepsilon$  for all  $|\alpha| \leq 1$

Hence  $x \perp_\varepsilon g$  for all  $\varepsilon > 0$

(c) Let  $x \in X$  and  $\varepsilon > 0$ , then  $\|0\| \leq \|0 + \alpha x\| + \varepsilon$ , and so  $0 \perp_\varepsilon x$ .

We have also  $\|x\| \leq \|x\| + \varepsilon$ , then  $\|x\| \leq \|x + \alpha 0\| + \varepsilon$ ,

Hence  $x \perp_\varepsilon 0$ .

(d) Let  $x \perp_\varepsilon g$ , and  $|\beta| < 1$ , then  $\|x\| \leq \|x + \alpha g\| + \varepsilon$ .

Multiply both sides by  $|\beta|$ ,

$$\text{we get } |\beta| \|x\| \leq |\beta| \|x + \alpha g\| + |\beta| \varepsilon$$

$$\leq |\beta| \|x + \alpha g\| + \varepsilon, \text{ and so}$$

$$\| \beta x \| \leq \| \beta x + \alpha \beta g \| + \varepsilon$$

Therefore,  $\beta x \perp_\varepsilon \beta g$

Hence the proof

R. Theorem 1.18

Let  $G$  be a subspace of a normed linear space  $X$ . If  $x \in X$ ,  $\varepsilon > 0$  and  $\delta \geq \varepsilon$ , then  $P_G(x, \varepsilon) \subset P_G(x, \delta)$ .

*Proof:*

Let  $g_0 \in P_G(x, \varepsilon)$ . Then by [Definition 1.7], we have

$$\|x - g_0\| \leq \|x - g\| + \varepsilon \text{ for all } g \in G \text{ and } \varepsilon > 0$$

$$\text{Then } \|x - g_0\| \leq \|x - g\| + \varepsilon \leq \|x - g\| + \delta$$

[since  $\delta > \varepsilon$ ], then,  $g_0 \in P_G(x, \delta)$ .

Therefore  $P_G(x, \varepsilon) \subset P_G(x, \delta)$

Hence the proof

## II. CONCLUSION

Here, I conclude my paper as  $\varepsilon$ -Best approximation and  $\varepsilon$ - orthogonality has the properties which are similar to the properties of best approximation.

## REFERENCE

- [1] Mazaheri .H ,Maalek Ghaini.F.m, ' Quasi –Orthogonality of the best approximation sets', Non linear Analysis ,65534-537(2006)
- [2] White.A.2-Banach Spacs,Math.Nachr,42(1969),43-60
- [3] Rivlin.Theodore J 'The Approximation of Function' 1969





10.22214/IJRASET



45.98



IMPACT FACTOR:  
7.129



IMPACT FACTOR:  
7.429



# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24\*7 Support on Whatsapp)