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The First Order Functional Differential Equation

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Abstract: In this paper, we discuss the first order functional differential equation for existence as well as for uniformly global attractivity of the solution. Here, we use classical hybrid fixed point theory to discuss quadratic functional differential equation on unbounded intervals.
Keywords: Functional differential equation; fixed point theorem, uniformly global attractivity.

I. INTRODUCTION

The fixed point theorem is core part of nonlinear functional analysis and originated in the works of Schauder[18], Banas[`18], Tarski[18], Deimling [6], Zeidler[18]. A fixed point theorems are useful for applications to other areas of mathematics such as theory of differential and integral equations, approximation and optimization theory, control theory, economics and game theory is classified as applicable fixed point theorems.

Most of applications of the fixed point theorems to nonlinear problems of any dynamical systems are existential in nature. However, now it is clear that the fixed point theorems is also useful in obtaining the different characterizations of the solutions. In Heikkila and Lakshmikantham[15], , Burton and Zhag[5], , Banas and Dhage[2], .The nature of nonlinearities involved in a differential equation, a fixed point theorem is used to prove the existence of solutions for equivalent operator equation which implies the existence results for the functional differential equations.

In this paper, we characterize the solutions of some nonlinear functional differential equations through applicable classical and hybrid fixed point theorems in abstract spaces. We claim that our results are new to the theory of nonlinear functional differential equations on unbounded intervals.

II. FUNCTIONAL DIFFERENTIAL EQUATIONS

In this article, we discuss three types of nonlinear functional differential equations on unbounded intervals of real line for existence as well as for some characterizations of the solutions via classical fixed point theorems in Banach spaces.

Let R be the real line and let R_+ be set of nonnegative real numbers. Let $I_0 = [-\delta, 0]$ be a closed and bounded interval in R for some real number $\delta > 0$ and let $J = I_0 \cup R_+$ Let C denote the Banach space of continuous real-valuied functions ϕ on I_0 with the supremum norm $\|.\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|$$

Clearly, C is a Banach space with this supremum norm. For a fixed $t \in R_+$, let x_t denote the element of C defined by

$$x_t(\theta) = x(t+\theta), \theta \in [-\delta, 0].$$

The space C is called the history space of the past interval I_0 for the functional differential equations to describing the past history of the problems in question.

Let $CRB(R_+)$ denote the class of functions $a: R_+ \to R - \{0\}$ satisfying the following properties:

1) a is continuous,

2) $\lim_{t\to\infty} a(t) = \pm\infty$, and

3) a(0) = 1.

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There do exist functions satisfying the above conditions. In fact, if $a_1(t)=t+1, a_2(t)=e^t$, , then $a_1, a_2 \in CRB(R_+)$. Again, the class of continuous and strictly monotone functions $a: R_+ \to R-\{0\}$ *with* $a(0)=1$ satisfy the above criteria. Note that if $a \in CRB(R_+)$, then the reciprocal function $\bar{a}: R_+ \to R$ defined by $\bar{a}:(t)=\frac{1}{a(t)}$ is continuous and $\lim_{t\to\infty} \bar{a}(t)=0.$

Consider the following functional differential equation,

$$\frac{d}{dt}[a(t)x(t)] = f(t, x(t), x_t) + g(t, x(t), x_t) \qquad a.e.\ t \in R_+ \qquad (2.1)$$

$$x_0 = \phi$$

Where, $a \in CRB(R_+)$ *and* $f, g: R_+ \times R \times C \to R.$ With given a function $\phi \in C$.

It is clear that the functional differential equation (FDE) (1.2.1)the scalar perturbations of second kind for the following nonlinear first order FDE on unbounded interval,

$$x(t) = g(t, x(t), x_t)\, a.e. t \in R_+ \qquad (2.2)$$
$$x_0 = \phi$$

The functional differential equation (2.1) are new to the theory of nonlinear differential equations and some special cases of these functional differential equations with $a \equiv 1$ have been studied in the literature on closed and bounded intervals for various aspects of the solutions. However, functional differential equations (2.1) is not discussed literature on unbounded intervals of real line.

In this paper, we discuss the above mentioned functional differential equations for existence as well as for different characterizations of the solutions such as attractivity, asymptotic attractivity and ultimate positivity of the solutions.

III. FIXED POINT THEORY

Let X be a nonempty set and let T : $X \to X$. An invariant point under T in X is called a fixed point of T, that is, the fixed points are the solutions of the functional equation T $x = x$. Any statement asserting the existence of fixed point of the mapping T is called fixed point theorem for the mapping T in X. The fixed point theorems are obtained by imposing the conditions on T or on X or on both T and X. By experience, better the mapping T or X, we have better fixed point principles. As we go on adding richer structure to the non-empty set X, we derive richer fixed point theorems useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations.

we give some fixed point theorems useful in establishing the attrratctivity and ultimate positivity of the solutions for functional differential equations (1.2.1) on unbounded intervals.

we give some basic part.

Let X be an infinite dimensional Banach space with the norm$\|.\|$. A mapping Q : $X \to X$ is called D-Lipschitz if there is a continuous and nondecreasing function $\phi: R_+ \to R_+$ satisfying

$$\| Qx - Qy\| \le \phi(\|x-y\|)$$

for all $x, y \in X$, where $\phi(0)=0.$ If $\phi(r)=k\,r,\ k>0,$ then Q is called Lipschitz with the Lipschitz constant k. In particular, if k<1, then Q is called a contraction on X with the contraction constant k. Further, if $\phi(r) < r$ *for* $r>0,$ then Q is called nonlinear D-contraction and the function $\emptyset$ is called D-function of Q on X. There do exist D-functions and the commonly used D-functions are $\phi(r)=k\,r$ *and* $\phi(r)=\frac{r}{1+r}$, etc.

A. *Main Result*

Theorem 3.1. (Granas and Dugundji) [13]**.** Let S be a non-empty, closed, convex and bounded subset of the Banach space X and let $Q: S \to S$ be a continuous and compact operator. Then the operator equation $Q\ x = x$ has a solution in S.

Theorem 3.2. (Dhage[10]). Let S be a closed, convex and bounded subset of the Banach space X and let $A: X \to X$ and $B: S \to X$ be two operators such that

1) A is nonlinear D-contraction,
2) B is completely continuous,
3) $x = Ax + By \Rightarrow x \in S$ *for all* $y \in S$.

Then the operator equation

$$Ax + Bx = x$$ has a solution in S.

Theorem 3.3. (Dhage[10]). Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and Let $A : X \to X$ *and* $B : S \to X$ be two operators such that

a) A is D-Lipschitz with D-function ψ,
b) B is completely continuous,
c) $x = Ax\ By \Rightarrow x \in S$ *for all* $y \in S$, *and*
d) $M\psi(t) < r$, *where* $M = \| B(S) \| = \sup\{\| Bx \| : x \in S\}$.

Then the operator equation

$$Ax\,Bx = x$$

has a solution in S.

A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [13], Deimling [6], Zeidler [18] and the references therein. In the following ,different types of characterizations of the solutions for nonlinear functional differential equations on unbounded intervals of real line.

IV. CHARACTERIZATIONS OF SOLUTIONS

We seek the solutions of the FDEs (2.1) in the space $BC(I_0 \cup R_+, R)$ of continuous and bounded real-valued functions defined on $I_0 \cup R_+$ Define a standard supremum norm $\| . \|$ and a multiplication " . " in BC $(I_0 \cup R_+, R)$ by

$$\| x \| = \sup_{t \in I_0 \cup R_+} | x(t) | \ \textit{and}\ (xy)(t) = x(t)y(t), \quad t \in R_+.$$

Clearly, BC $(I_0 \cup R_+, R)$ becomes a Banach algebra with respect to the above norm and the multiplication in it. By $L^1(R_+, R)$ we denote the space of lebesgue integrable functions on R_+ and the norm $\| . \|\ L^1$ *in* $L^1(R_+, R)$ is defined by

$$\| x \| L^1 = \int_0^\infty | x(t) | \, ds.$$

In order to introduce further concepts used in this paper, let us assume that E = $BC(I_0 \cup R_+, R)$ and let Ω be a non-empty subset of X. Let Q : $E \to E$ be a operator and consider the following operator equation in E

$$Qx(t) = x(t) \qquad (4.1)$$

for all $t \in I_0 \cup R_+$. Below we give different characterizations of the solutions for the operator equation (2.4.1) in the space $BC(I_0 \cup R_+, R)$

Definition 4.1. We say that solutions of the operator equation (2.4.1) are locally attractive if there exists a closed ball $\bar{B}_r(x_0)$ in the space $BC(I_0 \cup R_+, R)$ for some $x_0 \in BC(I_0 \cup R_+, R)$ such that for arbitrary solutions $x = x(t)$ *and* $y = y(t)$ of equation (4.1) belonging to $\bar{B}_r(x_0)$ we have that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0 \qquad (4.2)$$

In the case when the limit (4.2) is uniform with respect to the set $\bar{B}_r(x_0)$, i.e., when for each $\in > 0$ there exists T >0 such that

$$| x(t) - y(t)) \le \in \qquad (4.3)$$

for all $x, y \in \overline{B}_r(x_0)$ being solutions of (4.1) and for $t \geq T$, we will say that solutions of equation (4.1) are uniformly locally attractive on $I_0 \cup R_+$.

Definition.4.2. A solution $x = x(t)$ of equation (1.4.1) is said to be globally attractive if (4.2) holds for each solution $y = y(t)$ of (4.1) in $BC(I_0 \cup R_+, R)$. In other words, we may say that solutions of the equation (1.4.1) are globally attractive if for arbitrary solutions x(t) and y(t) of (2.4.1) in $BC(I_0 \cup R_+, R)$ The condition (1.4.2) is satisfied. In the case when the condition (4.2) is satisfied uniformly with respect to the space $BC(I_0 \cup R_+, R)$ i.e., if for every ∈>0 there exists T>0 such that the inequality (1.4.2) is satisfied for al x,y,∈ $BC(I_0 \cup R_+, R)$ being the solutions of (4.1) and for $i \geq T$, we will say that solutions of the equation (4.1) are uniformly globally attractive on $I_0 \cup R_+$.

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (4.1) in the space $BC(I_0 \cup R_+, R)$

Definition4.3 . A solution x of the equation (4.1) is called locally ultimately positive if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(I_0 \cup R_+, R)$ for some x_0 ,∈ $BC(I_0 \cup R_+, R)$ such that $x \in \overline{B}_r(x_0)$ and

$$\lim_{t\to\infty} \left[|x(t)| - x(t) \right] = 0. \qquad (4.4)$$

In the case when the limit (4.4) is uniform with respect to the solution set of the operator equation (1.4.1) in $BC(I_0 \cup R_+, R)$ i.e., when for each ∈ > 0 there exists T > 0 such that

$$\left\| \, |x(t)| - x(t) \, \right\| \leq \in \qquad (4.5)$$

for all x being solutions of (4.1) in $BC(I_0 \cup R_+, R)$ and for $t \geq T$, we will say that solutions of equation (4.1) are uniformly locally ultimately positive on R_+

Definition 4.4. A solution $x \in BC(I_0 \cup R_+, R)$ of the equation (1.4.1) is called globally ultimately positive if (1.4.4) is satisfied. In the case when the limit (1.4.5) is uniform with respect to the solution set of the operator equation (1.4.1) in $BC(I_0 \cup R_+, R)$ i.e., when for each ∈ > 0 there exists T > 0 such that (4.5) is satisfied for all x being solutions of (1.4.1) in $BC(I_0 \cup R_+, R)$ and for $t \geq T$, we will say that solutions of equation (4.1) are uniformly globally ultimately positive on $I_0 \cup R_+$

V. ATTRACTIVITY AND POSITIVITY RESULTS

In this section, we prove the global attractivity and positivity results for the FDE (1.2.1) on $I_0 \cup R_+$ under some suitable conditions. Let I be a closed interval in R and let $AC(I, R)$ be the space of functions which are defined and absolutely continuous on I. As every absolutely continuous function is continuous on I, we have that $AC(I, R) \subset C(I, R)$ However, converse implication may not hold. It is also known that if ∈ $AC(I, R)$, then it is almost everywhere differentiable on I. First we prove the global attractivity and ultimate positivity results for the FDE (2.1) on $I_0 \cup R_+$.

First we discuss the FDE (2.1) for attractivity characterization of the solutions on unbounded interval $I_0 \cup R_+$. We need the following definitions in the sequel.

Definition 5.1. By a solution for the functional differential equation (2.1) we mean a function $x \in BC(I_0 \cup R_+, R) \cap AC(R_+, R)$ such that

1) The function $t \mapsto a(t)x(t)$ is absolutely continuous on R_+, and
2) x satisfies the equations in (2.1),

Where $AC(R_+, R)$ is the space of absolutely continuous real-valued functions on right half real axis R_+.

Definition 5.2. A function $\alpha : R_+ \times R \times C \to R$ is called caratheodory if

a) $t \mapsto \alpha(t, x, y)$ is measurable for all $x \in R$ *and* $y \in C$, *and*

b) $(x, y) \mapsto \alpha(t, x, y)$ is continuous for all $t \in R_+$.

We need the following hypotheses .

(A_1). There exists a continuous function h : $R_+ \to R_+$ such that

$$| g(t, x, y) | \leq h(t) \ a.e. \ t \in R_+$$

for all $x \in R$ and $y \in C$. Moreover, we assume that $\lim_{t\to\infty} | \bar{a}(t) | \int_0^t h(s) ds = 0$

(A_2). $\phi(0) \geq 0$.

Theorem 5.1. Assume that the hypotheses (A_1) holds. Then the FDE (2.1) has a solution and solutions are uniformly globally attractive on $I_0 \cup R_+$.

Proof. Set $X = BC(I_0 \cup R_+, R)$.Define an operator Q on X by

$$Qx(t) = \begin{cases} \phi(0)\bar{a}(t) + \bar{a}(t)\left[\int_0^t f(s, x(s), x_s) ds + \int_0^t g(s, x(s), x_s) ds\right] & if \ t \in R_+ \\ \phi(t), & if \ t \in I_0 \end{cases} \quad (5.1)$$

We show that Q defines a mapping $Q : X \to X$. let $x \in X$ be arbitrary. Obviously, Qx is a continuous function on $I_0 \cup R_+$. We show that Qx is bounded on $I_0 \cup R_+$ Thus, if $t \in R_+$, then we obtain:

$$| Qx(t) | \leq | \phi(0) | \, | \bar{a}(t) | + | \bar{a}(t) | \int_0^t | f(s, x(s), x_s) + g(s, x(s), x_s) | ds$$

$$| Qx(t) | \leq | \phi(0) | \, \|\bar{a}\| + | \bar{a}(t) | \int_0^t h(s) ds.$$

Since $\lim_{t\to\infty} | \bar{a}(t) | \int_0^t h(s) ds = 0$, and the function $w : R_+ \to R$ defined by $w(t) = | \bar{a}(t) | \int_0^t h(s) ds$ is continuous, there is constant W > 0 such that

$$\sup_{t\geq 0} w(t) = \sup_{t\geq 0} | \bar{a}(t) | \int_0^t h(s) ds \leq W.$$

Therefore,

$$| Qx(t) | \leq | \phi(0) | \, \|\bar{a}\| + W \leq \|\bar{a}\| \|\phi\| + W$$

for all $t \in R_+$. Similarly, if $t \in I_0$ then $| Qx(t) | \leq \|\phi\|$. As a result, we have that

$$\|Qx\| \leq (\|\bar{a}\| + 1)\|\phi\| + W \quad (5.2)$$

for all $x \in X$ and therefore, Q maps X into X itself. Define a closed ball $\bar{B}_r(0)$ centered at origin of radius r, where $r = (\|\bar{a}\| + 1)\|\phi\|$ W. Clearly Q defines a mapping $Q : X \to \bar{B}_r(0)$ and in particular $Q : \bar{B}_r(0) \to \bar{B}_r(0)$. We show that Q satisfies all the conditions of Theorem 3.1. First, we show that Q is continuous on $\bar{B}_r(0)$. To do this, let us fix arbitrarily $\in > 0$ and let $\{x_n\}$ be a sequence of points in $\bar{B}_r(0)$ converging to a point $\in \bar{B}_r(0)$. Then we get

$$| (Qx_n)(t) - (Qx)(t) | \leq | \bar{a}(t) | \int_0^t \begin{bmatrix} | f(s, x_n(s), x_n(\theta + s)) - f(s, x(s), x(\theta + s)) | - \\ | g(s, x_n(s), x_n(\theta + s)) - g(s, x(s), x(\theta + s)) | \end{bmatrix} | ds$$

$$\leq |\bar{a}(t)| \int_0^t \begin{Bmatrix} |[f(s,x_n(s),x_n(\theta+s)) - f(s,x(s),x(\theta+s))]| - \\ |[g(s,x_n(s),x_n(\theta+s)) - g(s,x(s),x(\theta+s))]| \end{Bmatrix} |ds$$

$$\leq 2|\bar{a}(t)| \int_0^t h(s)ds$$

$$\leq 2w(t) \tag{5.3}$$

Hence, by virtue of hypothesis (A_1), we infer that there exists a T >0 such that $w(t) \leq \in$ at $t \geq T$. Thus, for $t \geq T$ from the estimate (5.2) we derive that

$$|(Qx_n)(t) - (Qx)(t)| \leq 2\in \quad as \quad n \to \infty$$

Furthermore, let us assume that $t \in [0,T]$. Then, following arguments similar to those given in Dhage [2] and Ntouyas [17], by Lebesgue dominated convergence theorem, we obtain the estimate:

$$\lim_{n\to\infty} Qx_n(t) = \lim_{n\to\infty}\left[\phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t [f(s,x_n(s),x_n(\theta+s)) + g(s,x_n(s),x_n(\theta+s)]ds|\right]$$

$$= \phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t \left[\lim_{n\to\infty}[f(s,x_n(s),x_n(\theta+s)) + g(s,x_n(s),x_n(\theta+s)]ds\right] = Qx(t)$$

for all $t \in [0,T]$.similarly, if $t \in I_0$ then

$$\lim_{n\to\infty} Qx_n(t) = \phi(t) = Qx(t)$$

Thus, $Qx_n \to Qx \, as \, n \to \infty$ uniformly on R_+ and hence Q is a continuous operator on $\bar{B}_r(0)$ into $\bar{B}_r(0)$.

Next, we show that B is compact operator on $\bar{B}_r(0)$. To finish this, it is enough to show that every sequence $\{Qx_n\}$ in $Q(\bar{B}_r(0))$ has a cauchy subsequence. Now,

$$|Qx_n(t)| \leq |\phi(0)|\,|\bar{a}(t)| + |\bar{a}(t)|\int_0^t |f(s,x_n(s),x_n(\theta+s)) + g(s,x_n(s),x_n(\theta+s)|ds$$

$$\leq (\|\bar{a}\|+1)|\phi(0)| + w(t) \tag{5.4}$$

$$\leq (\|\bar{a}\|+1)|\,\|\phi\| + w(t)$$

For all $t \in R_+$ .Taking supremum over t, we obtain

$$\|Qx_n\| \leq (\|\bar{a}\|+1)\|\phi\| + W$$

for all $n \in N$. This shows that $\{Qx_n\}$ is a uniformly bounded sequence in $Q(\bar{B}_r(0))$.

Next, we show that $Q(\bar{B}_r(0))$ is also an equicontinuous set in X. Let $\in > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T_1 > 0$ such that $w(t) \leq \frac{\in}{8}$ for all $t \geq T_1$. Similarly, since $\lim_{t\to\infty} \bar{a}(t) = 0,$ for above $\in > 0$ there is a real number $T_2 > 0$ such that $|\bar{a}(t)| < \frac{\in}{8|\phi(0)|}$ for all $t \geq T$. Thus, if $T = \max\{T_1, T_2\}$, $then$ $|w(t)| < \frac{\epsilon}{8}$ and $|\bar{a}(t)| < \frac{\epsilon}{8|\phi(0)|}$ $for\ all\ t \geq T. Let$ $t,\tau \in I_0 \cup R_+$, be arbitrary, if $t,\tau \in I_0$ then by uniform countinuity of ϕ on I_0 ,for above $\in$ we have a $\delta_1 > 0$ which is a function of only $\in$ such that

$$|t-\tau| < \delta_1 \Rightarrow |Qx_n(t) - Qx_n(\tau)| = |\phi(t) - \phi(\tau)| < \frac{\in}{4}$$

for all $n \in N. If\ t,\tau \in [0,T]$, then we have

$$
\begin{aligned}
|Qx_n(t)-Qx_n(\tau)| \le & |\phi(0)||\bar{a}(t)-\bar{a}(\tau)| \\
& \left||\bar{a}(t)|\int_0^t f(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^\tau f(s,x_n(s),x_n(\theta+s))\,ds\right|+ \\
& \left||\bar{a}(t)|\int_0^t g(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^\tau g(s,x_n(s),x_n(\theta+s))\,ds\right| \\
\le & |\phi(0)||\bar{a}(t)-\bar{a}(\tau)| \\
& +\left|\bar{a}(t)\int_0^t f(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^t f(s,x_n(s),x_n(\theta+s))\,ds\right| \\
& +\left|\bar{a}(\tau)\int_0^t f(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^\tau f(s,x_n(s),x_n(\theta+s))\,ds\right| \\
& +\left|\bar{a}(t)\int_0^t g(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^t g(s,x_n(s),x_n(\theta+s))\,ds\right| \\
& +\left|\bar{a}(\tau)\int_0^t g(s,x_n(s),x_n(\theta+s))ds-\bar{a}(\tau)\int_0^\tau g(s,x_n(s),x_n(\theta+s))\,ds\right| \\
\le & |\phi(0)||\bar{a}(t)-\bar{a}(\tau)|+|\bar{a}(t)-\bar{a}(\tau)|\left|\int_0^t f(s,x_n(s),x_n(\theta+s))\,ds\right| \\
& +|\bar{a}(\tau)|\left|\int_\tau^t f(s,x_n(s),x_n(\theta+s))\,ds\right|+|\bar{a}(t)-\bar{a}(\tau)|\left|\int_0^t g(s,x_n(s),x_n(\theta+s))\,ds\right| \\
& +|\bar{a}(\tau)|\left|\int_\tau^t g(s,x_n(s),x_n(\theta+s))\,ds\right| \\
\le & |\phi(0)||\bar{a}(t)-\bar{a}(\tau)|+|\bar{a}(t)-\bar{a}(\tau)|\left|\int_0^T h(s)ds+\|\bar{a}\|\right|\int_\tau^t h(s)\,ds\Big| \\
\le & |\phi(0)||\bar{a}(t)-\bar{a}(\tau)|+|\bar{a}(t)-\bar{a}(\tau)|\left|\int_0^T h(s)ds+\|\bar{a}\|\right||p(t)-p(\tau)| \\
\le & \left[|\phi(0)|+\|h\|_{L^1}\right]|\bar{a}(t)-\bar{a}(\tau)|+\|\bar{a}\||p(t)-p(\tau)|
\end{aligned}
$$

Where, $p(t)=\int_0^t h(s)ds \;\; and \; \|h\|_{L^1}=\int_0^\infty h(s)ds.$

By the uniform continuity of the functions $\bar{a}$ and p on [0,T], for above $\in$ we have the real numbers $\delta_2 > 0 \; and \; \delta_3 > 0$ which are the functions of only $\in$ such that

$$|t-\tau|<\delta_2 \Rightarrow |\bar{a}(t)-\bar{a}(\tau)|<\frac{\in}{8\left[|\phi(0)|+\|h\|_{L^1}\right]}$$

and

$$|t-\tau|<\delta_3 \Rightarrow |p(t)-p(\tau)|<\frac{\in}{8\|\bar{a}\|}$$

$Let \; \delta_4 = \min\{\delta_2,\delta_3\}$.Then

$$|t-\tau|<\delta_4 \Rightarrow |Qx_n(t)-Qx_n(\tau)|<\frac{\in}{4}$$

for all $n \in N$. similarly, if $t \in I_0 and \; \tau \in [0,T], then$

$$|Qx_n(t)-Qx_n(\tau)|\le |Qx_n(t)-Qx_n(0)|+|Qx_n(0)-Qx_n(\tau)|$$

Take $\delta_5 = \min\{\delta_1,\delta_4\} > 0$ which is again a function of only $\in$. Hence by above estimated facts it follows that

$$|t-\tau|<\delta_5 \Rightarrow |Qx_n(t)-Qx_n(\tau)|<\frac{\in}{2}$$

for all $n \in N$

Again, if $t, \tau > T$, then we have a real number $\delta_6 > 0$ which is a function of only $\in$ such that

$$|Qx_n(t) - Qx_n(\tau)| \leq |\phi(0)||a(t) - a(\tau)|$$
$$+\left|\bar{a}(t)\int_0^t f(s, x_n(s), x_n(\theta+s))\,ds - \bar{a}(t)\int_0^\tau f(s, x_n(s), x_n(\theta+s))\,ds\right|$$
$$+\left|\bar{a}(t)\int_0^t g(s, x_n(s), x_n(\theta+s))\,ds - \bar{a}(t)\int_0^\tau g(s, x_n(s), x_n(\theta+s))\,ds\right|$$
$$\leq |\phi(0)||a(t)| + |\phi(0)||a(\tau)| + w(t) + w(\tau)$$
$$\leq \frac{\in}{4} + \frac{\in}{4}$$

For all $n \in N$, whenever $t - \tau| < \delta_6$. $similarly, if\ t, \tau \in I_0 \cup R_+$ with $< T < \tau$, then we have

$$|Qx_n(t) - Qx_n(\tau)| \leq |Qx_n(t) - Qx_n(T)| + |Qx_n(T) - Qx_n(\tau)|.$$

Take $\delta = \min\{\delta_5, \delta_6\} > 0$ which is again a function of only $\in$. Therefore, from the above obtained estimates, it follows that

$$|Qx_n(t) - Qx_n(T)| < \frac{\in}{2}\ and\ |Qx_n(T) - Qx_n(\tau)| < \frac{\in}{2}$$

For all $n \in N$, whenever $t - \tau| < \delta$. As a result, $Qx_n(t) - Qx_n(\tau)| < \in$ for all $t, \tau \in I_0 \cup R_+\ and\ for\ all\ n \in N, whenever |t - \tau| < \partial$. This shows that $\{Qx_n\}$ is equicontinuous sequence in X. Now an application of Arzela-Ascoli theorem yields that $\{Qx_n\}$ has a uniformly convergent subsequence on the compact subset $I_0 \cup [0, T]$ of $I_0 \cup R$ .Without loss of generality, call the subsequence to be the sequence itself.

We show that $\{Qx_n\}$ is Cauchy in X. Now $|Qx_n(t) - Qx(t)| \to 0\ as\ n \to \infty\ for\ all\ t \in I_0 \cup [0, T]$. then for given $\in > 0$ there exits an $n_0 \in N$ such that

$$\sup_{-\delta \leq p \leq T} |\bar{a}(p)| \left[\int_0^p |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))\right] +$$
$$\left[\int_0^p |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))\right]|\,ds < \frac{\in}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$\|Qx_m - Qx_n\| = \sup_{-\delta \leq t < \infty} \|\bar{a}(t)| \left[\begin{array}{l}\int_0^t |f(s, x_m(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))\,ds| \\ +\int_0^t |g(s, x_m(s), x_m(\theta+s)) - g(s, x_n(s), x_n(\theta+s))\,ds|\end{array}\right]$$

$$\leq \sup_{-\delta \leq p \leq T} \|\bar{a}(p)| \left[\begin{array}{l}\int_0^p |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))\,ds| + \\ |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))\,ds|\end{array}\right]$$
$$+ \sup_{p \geq T} \|\bar{a}(p)| \left[\begin{array}{l}\int_0^p |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))\,ds| + \\ |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))\,ds|\end{array}\right]$$
$$\leq \in.$$

This shows that $\{Qx_n\} \subset Q(\overline{B_r}(0)) \subset X$ is Cauchy. Since X is complete, $\{Qx_n\}$ converges to a point in X. As $Q(\overline{B_r}(0))$ is closed $\{Qx_n\}$ converges to a point in $Q(\bar{B}_r(0))$. Hence $Q(\bar{B}_r(0))$ is relatively compact and consequently Q is a continuous and compact operator on $\bar{B}_r(0)$ into itself, Now an application of theorem 3.2 to the operator Q on $\overline{B_r}(0)$ `yields that Q has a fixed point in $\overline{B_r}(0)$ which further implies that the FDE (2.1) has a solution defined on $I_0 \cup R_+$.

Finally, we show that the solutions are uniformly attravtive on $I_0 \cup R_+$. Let $x, y \in \bar{B}_r(0)$ be any two solutions the FDE (4.1) defined on $I_0 \cup R_+$. Then,

$$|x(t)-y(t)| \leq \left|[\bar{a}(t)\int_0^t f(s,x(s),x_s)ds - \bar{a}(t)\int_0^t f(s,y(s),y_s)ds] + [\bar{a}(t)\int_0^t g(s,x(s),x_s)ds - \bar{a}(t)\int_0^t g(s,y(s),y_s)ds]\right|$$

$$\leq |\bar{a}(t)|\int_0^t \left|f(s,x(s),x_s)\right|ds + |\bar{a}(t)|\int_0^t \left|f(s,x(s),x_s)\right|ds +$$

$$|\bar{a}(t)|\int_0^t \left|g(s,x(s),x_s)\right|ds + |\bar{a}(t)|\int_0^t \left|g(s,x(s),x_s)\right|ds$$

$$\leq 2w(t) \qquad (1.5.5)$$

For all $t \in I_0 \cup R_+$. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number T>0 such that $w(t) < \frac{\epsilon}{2}$ $for\ all\ t \geq T. therefore,$ $|x(t) - y(t)| \leq \epsilon$ $for\ all\ t \geq T$, and so all the solutions of the FDE (2.2.1) are uniformly globally attractive on $I_0 \cup R_+$.

Theorem 5.2. Assume that the hypotheses $(A_1) - (A_2)$ hold. Then the FDE (2.1) has a solution and solutions are uniformly globally attractive and ultimately positive on $I_0 \cup R_+$.

Proof. By Theorem 5.1, the FDE (1.2.1) has a solution in $\overline{B_r}(0)$, $where\ r = \|\emptyset\| + W$ and the solutions are uniformly globally attractive on $I_0 \cup R_+$. We know that for any $x, y \in R$, one has the inequality.

$$|x|+|y| \geq |x+y| \geq x+y,$$

And, therefore,

$$\| x+y|-(s+y)| \leq || x|+|y|-(x+y)| \leq \| x|-x| + \| y|-y| \qquad (5.6)$$

for all $x, y \in R$. Now for any solution $x \in \overline{B_r}(0)$, one has

$$\| x(t)|-x(t)| = \| \phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t f(s,x(s),x_s)ds| - (\phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t f(s,x(s),x_s)ds)| +$$

$$|\phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t g(s,x(s),x_s)ds| - (\phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t g(s,x(s),x_s)ds)|$$

$$\leq \| \phi(0)| - \phi(0)\| \bar{a}(t)| + |\bar{a}(t)|\int_0^t g(s,x(s),x_s)|ds + |\bar{a}(t)|\int_0^t g(s,x(s),x_s)|ds +$$

$$|\bar{a}(t)|\int_0^t g(s,x(s),x_s)|ds + |\bar{a}(t)|\int_0^t g(s,x(s),x_s)|ds$$

$$\leq 2w(t).$$

Since $\lim_{t\to\infty} w(t) = 0$, there is a real number T > 0 such that $\||x(t)| - \leq \epsilon$ for all $t \geq T$. Hence solutions of the FDE (2.1) are also uniformly globally ultimately Positive on $I_0 \cup R_+$. This completes the proof.

VI. EXAMPLE

Let $I_0 = [-\frac{\pi}{2}, 0]$ be a closed and bounded interval in R and define a function $\emptyset : I_0 \to R$ by $\emptyset(t) = cost$. consider the following FDE,

$$(e^t x(t))' = e^{-t}\frac{(x(t)+x_t)}{|x(t)|+\|x_t\|c} + e^{-t}\frac{x(t)}{|x(t)|+\|x_t\|c} \quad a.e.t \in R_+$$

$$x_0 = \phi,$$

Where, $e^{-t} \in C(R_+,R) \subset L^1(R_+,R)$ $and \lim_{t\to\infty} e^{-t}\int_0^t e^{-s}ds = 0$

Here, $a(t) = e^t$ which is positive and increasing on R_+ and so $a \in CRB(R_+)$ and

$$\|\bar{a}\| = \sup_{t\geq 0}\bar{a}(t) = \sup_{t\geq 0} e^{-t} \leq 1.$$

Again, $f(t,x,y) = \dfrac{e^{-t}(x+y)}{|x|+\|y\|c}$, $g(t,x,y) = e^{-t}\dfrac{x(t)}{|x(t)|+\|x_t\|c}$ $for\ t \in R_+, x \in R\ and\ y \in C$. Clearly, the function g satisfies the hypothesis (A_1) with growth function $h(t) = e^{-t}\ on\ R_+\ so\ that \lim_{t\to\infty} w(t) = \lim_{t\to\infty} e^{-t}\int_0^t e^{-s}ds = 0$. Now we apply theorem 5.1. to

FDE (2.1) to conclude that it has solution and solutions are uniformly globally attractive on $I_0 \cup R_+$. As $\phi(0)=1\geq 0$, , *the hypothesis* (A_2) of Theorem 1.5.2. is satisfied. Hence, solutions of the given FDE are also uniformly globally ultimately positive on $I_0 \cup R_+$.

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