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International Journal For Research in  
Applied Science and Engineering Technology



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# **INTERNATIONAL JOURNAL FOR RESEARCH**

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

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**Volume: 7      Issue: 1      Month of publication: January 2019**

**DOI: <http://doi.org/10.22214/ijraset.2019.1041>**

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# On the Positive Pell Equation $y^2 = 35x^2 + 14$

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**Abstract:** The binary quadratic equation represented by the positive Pellian  $y^2 = 35x^2 + 14$  is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

**Keywords:** Binary quadratic, hyperbola, integral solutions, parabola, pell equation. 2010 mathematics subject classification: 11D09

## I. INTRODUCTION

A binary quadratic equation of the form  $y^2 = Dx^2 + 1$  where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, yet another interesting hyperbola given by  $y^2 = 35x^2 + 14$  is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

### A. Method of Analysis

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 + 14 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 7$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 35x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{35}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (6 + \sqrt{35})^{n+1} + (6 - \sqrt{35})^{n+1}$$

$$g_n = (6 + \sqrt{35})^{n+1} - (6 - \sqrt{35})^{n+1}, n = -1, 0, 1, 2, \dots$$

Applying Brahmagupta lemma between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$ , the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{35}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{\sqrt{35}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 12x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 12y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical examples

$n$	$x_{n+1}$	$y_{n+1}$
-1	1	7
0	13	77
1	155	917
2	1847	10927
3	22009	130207

From the above table, we observe some interesting relations among the solutions which are presented below:

- 1)  $x_{n+1}$  and  $y_{n+1}$  are always odd.
- 2) Relations among the solutions
  - a)  $x_{n+2} - 6x_{n+1} - y_{n+1} = 0$
  - b)  $y_{n+2} - 6x_{n+2} + x_{n+1} = 0$
  - c)  $y_{n+3} - 71x_{n+2} - 6x_{n+1} = 0$
  - d)  $x_{n+3} - 12x_{n+2} + x_{n+1} = 0$
  - e)  $x_{n+3} - 71x_{n+2} - y_{n+1} = 0$
  - f)  $x_{n+3} - 6x_{n+2} - y_{n+2} = 0$
  - g)  $x_{n+3} - 71x_{n+1} - 12y_{n+1} = 0$
  - h)  $35x_{n+3} - 6y_{n+3} + y_{n+2} = 0$
  - i)  $y_{n+1} - x_{n+2} + 6x_{n+1} = 0$
  - j)  $12y_{n+1} - x_{n+3} - x_{n+1} = 0$
  - k)  $12y_{n+1} - x_{n+3} + 71x_{n+1} = 0$
  - l)  $2y_{n+2} - x_{n+3} + x_{n+1} = 0$
  - m)  $12y_{n+3} - 71x_{n+3} + x_{n+1} = 0$
  - n)  $349x_{n+1} - 72y_{n+1} + x_{n+3} = 0$
  - o)  $349x_{n+2} - 781y_{n+1} + 6x_{n+3} = 0$
  - p)  $349y_{n+2} - 4614y_{n+1} + 35x_{n+3} = 0$
  - q)  $349y_{n+3} - 55019y_{n+1} + 420x_{n+3} = 0$
  - r)  $6y_{n+1} - 71y_{n+2} + 35x_{n+3} = 0$
  - s)  $6y_{n+3} - 35x_{n+3} - y_{n+2} = 0$
  - t)  $y_{n+1} - 71y_{n+3} + 420x_{n+3} = 0$
  - u)  $y_{n+3} - 12y_{n+2} + y_{n+1} = 0$
  - v)  $280x_{n+1} - 3y_{n+3} + 353y_{n+1} = 0$
  - w)  $y_{n+2} - 35x_{n+1} - 6y_{n+1} = 0$
  - x)  $y_{n+3} - 420x_{n+1} - 71y_{n+1} = 0$
  - y)  $6y_{n+3} - 35x_{n+1} - 71y_{n+2} = 0$

$$z) \quad y_{n+1} - 6x_{n+3} + 71x_{n+2} = 0$$

$$aa) \quad y_{n+2} - x_{n+3} + 6x_{n+2} = 0$$

$$bb) \quad y_{n+3} - 6x_{n+3} + x_{n+2} = 0$$

$$cc) \quad y_{n+3} - 70x_{n+2} - y_{n+1} = 0$$

$$dd) \quad 6y_{n+2} - 35x_{n+1} - y_{n+1} = 0$$

$$ee) \quad y_{n+3} - 35x_{n+2} + 6y_{n+2} = 0$$

3) Each of the following expressions represents a nasty number

$$a) \quad 6x_{2n+3} - 66x_{2n+2} + 12$$

$$b) \quad \frac{1}{2}(x_{2n+4} - 131x_{2n+2} + 24)$$

$$c) \quad 6y_{2n+2} - 30x_{2n+2} + 12$$

$$d) \quad y_{2n+3} - 65x_{2n+2} + 12$$

$$e) \quad \frac{1}{71}(6y_{2n+4} - 4650x_{2n+2} + 852)$$

$$f) \quad 66x_{2n+4} - 786x_{2n+3} + 12$$

$$g) \quad 11y_{2n+2} - 5x_{2n+3} + 12$$

$$h) \quad 66y_{2n+3} - 390x_{2n+3} + 12$$

$$i) \quad 11y_{2n+4} - 775x_{2n+3} + 12$$

$$j) \quad \frac{6}{71}(131y_{2n+2} - 5x_{2n+4} + 142)$$

$$k) \quad 131y_{2n+3} - 65x_{2n+4} + 12$$

$$l) \quad 786y_{2n+4} - 4650x_{2n+4} + 12$$

$$m) \quad \frac{1}{7}(78y_{2n+2} - 6y_{2n+3} + 84)$$

$$n) \quad \frac{1}{84}(930y_{2n+2} - 6y_{2n+4} + 1008)$$

$$o) \quad \frac{1}{7}(930y_{2n+3} - 78y_{2n+4} + 84)$$

4) Each of the following expressions represents a cubical integer

$$a) \quad x_{3n+4} - 11x_{3n+3} - 33x_{n+1} + 3x_{n+2}$$

$$b) \quad \frac{1}{12}[x_{3n+5} - 131x_{3n+3} - 393x_{n+1} + 3x_{n+3}]$$

$$c) \quad y_{3n+3} - 5x_{3n+3} - 15x_{n+1} + 3y_{n+1}$$

$$d) \quad \frac{1}{6}[y_{3n+4} - 65x_{3n+3} - 195x_{n+1} + 3y_{n+2}]$$

$$e) \frac{1}{71} [y_{3n+5} - 775x_{3n+3} - 2325x_{n+1} + 3y_{n+3}]$$

$$f) 11x_{3n+5} - 131x_{3n+4} - 393x_{n+2} + 33x_{n+3}$$

$$g) \frac{1}{6} [11y_{3n+3} - 5x_{3n+4} - 15x_{n+2} + 33y_{n+1}]$$

$$h) 11y_{3n+4} - 65x_{3n+4} - 195x_{n+2} + 33y_{n+2}$$

$$i) \frac{1}{6} [11y_{3n+5} - 775x_{3n+4} - 2325x_{n+2} + 33y_{n+3}]$$

$$j) \frac{1}{497} [917y_{3n+3} - 35x_{3n+5} + 2751y_{n+1} - 105x_{n+3}]$$

$$k) \frac{1}{6} [131y_{3n+4} - 65x_{3n+5} - 195x_{n+3} + 393y_{n+2}]$$

$$l) 131y_{3n+5} - 775x_{3n+5} - 2325x_{n+3} + 393y_{n+3}$$

$$m) \frac{1}{7} [13y_{3n+3} - y_{3n+4} + 39y_{n+1} - 3y_{n+2}]$$

$$n) \frac{1}{84} [155y_{3n+3} - y_{3n+5} + 465y_{n+1} - 3y_{n+3}]$$

$$o) \frac{1}{7} [155y_{3n+4} - 13y_{3n+5} + 465y_{n+2} - 39y_{n+3}]$$

5) Each of the following expressions represents a bi-quadratic integer

$$a) x_{4n+5} - 11x_{4n+4} - 44x_{2n+2} + 4x_{2n+3} + 6$$

$$b) \frac{1}{12} [x_{4n+6} - 131x_{4n+4} - 524x_{2n+2} + 4x_{2n+4} + 72]$$

$$c) y_{4n+4} - 5x_{4n+4} - 20x_{2n+2} + 4y_{2n+2} + 6$$

$$d) \frac{1}{6} [y_{4n+5} - 65x_{4n+4} - 260x_{2n+2} + 4y_{2n+3} + 36]$$

$$e) \frac{1}{71} [y_{4n+6} - 775x_{4n+4} - 3100x_{2n+2} + 4y_{2n+4} + 426]$$

$$f) 11x_{4n+6} - 131x_{4n+5} - 524x_{2n+3} + 44x_{2n+4} + 6$$

$$g) \frac{1}{6} [11y_{4n+4} - 5x_{4n+5} - 20x_{2n+3} + 44y_{2n+2} + 36]$$

$$h) 11y_{4n+5} - 65x_{4n+5} - 260x_{2n+3} + 44y_{2n+3} + 6$$

$$i) \frac{1}{6} [11y_{4n+6} - 775x_{4n+5} - 3100x_{2n+3} + 44y_{2n+4} + 36]$$

$$j) \frac{1}{71} [131y_{4n+4} - 5x_{4n+6} + 524y_{2n+2} - 20x_{2n+4} + 426]$$

$$k) \frac{1}{6} [131y_{4n+5} - 65x_{4n+6} - 260x_{2n+4} + 524y_{2n+3} + 36]$$

$$l) 131y_{4n+6} - 775x_{4n+6} - 3100x_{2n+4} + 524y_{2n+4} + 6$$

$$m) \frac{1}{7}[13y_{4n+4} - y_{4n+5} + 52y_{2n+2} - 4y_{2n+3} + 42]$$

$$n) \frac{1}{84}[155y_{4n+4} - y_{4n+6} + 620y_{2n+2} - 4y_{2n+4} + 504]$$

$$o) \frac{1}{7}[155y_{4n+5} - 13y_{4n+6} + 620y_{2n+3} - 52y_{2n+4} + 42]$$

6) Each of the following expressions represents a quintic integer

$$a) x_{5n+6} - 11x_{5n+5} - 55x_{3n+3} + 5x_{3n+4} - 110x_{n+1} + 10x_{n+2}$$

$$b) \frac{1}{12}[x_{5n+7} - 131x_{5n+5} - 655x_{3n+3} + 5x_{3n+5} - 1310x_{n+1} + 10x_{n+3}]$$

$$c) y_{5n+5} - 5x_{5n+5} - 25x_{3n+3} + 5y_{3n+3} - 50x_{n+1} + 10y_{n+1}$$

$$d) \frac{1}{6}[y_{5n+6} - 65x_{5n+5} - 325x_{3n+3} + 5y_{3n+4} - 650x_{n+1} + 10y_{n+2}]$$

$$e) \frac{1}{71}[y_{5n+7} - 775x_{5n+5} - 3875x_{3n+3} + 5y_{3n+5} - 7750x_{n+1} + 10y_{n+3}]$$

$$f) 11x_{5n+7} - 131x_{5n+6} - 655x_{3n+4} + 55x_{3n+5} - 1310x_{n+2} + 110x_{n+3}$$

$$g) \frac{1}{6}[11y_{5n+5} - 5x_{5n+6} - 25x_{3n+4} + 55y_{3n+3} - 50x_{n+2} + 110y_{n+1}]$$

$$h) 11y_{5n+6} - 65x_{5n+6} - 325x_{3n+4} + 55y_{3n+4} - 650x_{n+2} + 110y_{n+2}$$

$$i) \frac{1}{6}[11y_{5n+7} - 775x_{5n+6} - 3875x_{3n+4} + 55y_{3n+5} - 7750x_{n+2} + 110y_{n+3}]$$

$$j) \frac{1}{71}[655y_{3n+3} - 524y_{5n+5} - 20x_{5n+7} - 25x_{3n+5} + 1965y_{n+1} - 75x_{n+3}]$$

$$k) \frac{1}{6}[131y_{5n+6} - 65x_{5n+7} - 325x_{3n+5} + 655y_{3n+4} - 650x_{n+3} + 1310y_{n+2}]$$

$$l) 131y_{5n+7} - 775x_{5n+7} - 3875x_{3n+5} + 655y_{3n+5} - 7750x_{n+3} + 1310y_{n+3}$$

$$m) \frac{1}{7}[13y_{5n+5} - y_{5n+6} + 65y_{3n+3} - 5y_{3n+4} + 130y_{n+1} - 10y_{n+2}]$$

$$n) \frac{1}{84}[155y_{5n+5} - y_{5n+7} + 775y_{3n+3} - 5y_{3n+5} + 1550y_{n+1} - 10y_{n+3}]$$

$$o) \frac{1}{7}[155y_{5n+6} - 13y_{5n+7} + 775y_{3n+4} - 65y_{3n+5} + 1550y_{n+2} - 130y_{n+3}]$$

### B. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

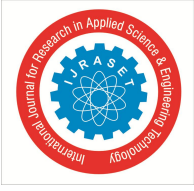
S. No	Hyperbolas	(X, Y)
1	$7X^2 - 5Y^2 = 28$	$(x_{n+2} - 11x_{n+1}, 13x_{n+1} - x_{n+2})$
2	$7X^2 - 5Y^2 = 4032$	$(x_{n+3} - 131x_{n+1}, 155x_{n+1} - x_{n+3})$
3	$7X^2 - 5Y^2 = 28$	$(y_{n+1} - 5x_{n+1}, 7x_{n+1} - y_{n+1})$
4	$7X^2 - 5Y^2 = 1008$	$(y_{n+2} - 65x_{n+1}, 77x_{n+1} - y_{n+2})$
5	$7X^2 - 5Y^2 = 141148$	$(y_{n+3} - 775x_{n+1}, 917x_{n+1} - y_{n+3})$
6	$7X^2 - 5Y^2 = 28$	$(11x_{n+3} - 131x_{n+2}, 155x_{n+2} - 13x_{n+3})$
7	$7X^2 - 5Y^2 = 1008$	$(11y_{n+1} - 5x_{n+2}, 7x_{n+2} - 13y_{n+1})$
8	$7X^2 - 5Y^2 = 28$	$(11y_{n+2} - 65x_{n+2}, 77x_{n+2} - 13y_{n+2})$
9	$7X^2 - 5Y^2 = 1008$	$(11y_{n+3} - 775x_{n+2}, 917x_{n+2} - 13y_{n+3})$
10	$7X^2 - 5Y^2 = 141148$	$(131y_{n+1} - 5x_{n+3}, 7x_{n+3} - 155y_{n+1})$
11	$7X^2 - 5Y^2 = 1008$	$(131y_{n+2} - 65x_{n+3}, 77x_{n+3} - 155y_{n+2})$
12	$7X^2 - 5Y^2 = 28$	$(131y_{n+3} - 775x_{n+3}, 917x_{n+3} - 155y_{n+3})$
13	$5X^2 - 7Y^2 = 980$	$(13y_{n+1} - y_{n+2}, y_{n+2} - 11y_{n+1})$
14	$5X^2 - 7Y^2 = 141120$	$(155y_{n+1} - y_{n+3}, y_{n+3} - 131y_{n+1})$
15	$5X^2 - 7Y^2 = 980$	$(155y_{n+2} - 13y_{n+3}, 11y_{n+3} - 131y_{n+2})$

2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X, Y)
1	$7X - 5Y^2 = 14$	$(x_{2n+3} - 11x_{2n+2}, 13x_{n+1} - x_{n+2})$
2	$84X - 5Y^2 = 2016$	$(x_{2n+4} - 131x_{2n+2}, 155x_{n+1} - x_{n+3})$
3	$7X - 5Y^2 = 14$	$(y_{2n+2} - 5x_{2n+2}, 7x_{n+1} - y_{n+1})$
4	$42X - 5Y^2 = 504$	$(y_{2n+3} - 65x_{2n+2}, 77x_{n+1} - y_{n+2})$
5	$497X - 5Y^2 = 70574$	$(y_{2n+4} - 775x_{2n+2}, 917x_{n+1} - y_{n+3})$
6	$7X - 5Y^2 = 14$	$(11x_{2n+4} - 131x_{2n+3}, 155x_{n+2} - 13x_{n+3})$
7	$42X - 5Y^2 = 504$	$(11y_{2n+2} - 5x_{2n+3}, 7x_{n+2} - 13y_{n+1})$
8	$7X - 5Y^2 = 14$	$(11y_{2n+3} - 65x_{2n+3}, 77x_{n+2} - 13y_{n+2})$
9	$42X - 5Y^2 = 504$	$(11y_{2n+4} - 775x_{2n+3}, 917x_{n+2} - 13y_{n+3})$
10	$497X - 5Y^2 = 70574$	$(131y_{2n+2} - 5x_{2n+4}, 7x_{n+3} - 155y_{n+1})$
11	$42X - 5Y^2 = 504$	$(131y_{2n+3} - 65x_{2n+4}, 77x_{n+3} - 155y_{n+2})$
12	$7X - 5Y^2 = 14$	$(131y_{2n+4} - 775x_{2n+4}, 917x_{n+3} - 155y_{n+3})$
13	$5X - Y^2 = 70$	$(13y_{2n+2} - y_{2n+3}, y_{n+2} - 11y_{n+1})$
14	$60X - Y^2 = 10080$	$(155y_{2n+2} - y_{2n+4}, y_{n+3} - 131y_{n+1})$
15	$5X - Y^2 = 70$	$(155y_{2n+3} - 13y_{2n+4}, 11y_{n+3} - 131y_{n+2})$





### REFERENCES

- [1] Dickson L.E., History of Theory of Numbers, Chelsea Publishing Company, New York, Vol. II, 1952
- [2] Mordell L.J., Diophantine Equations, Academic Press, New York, 1969.
- [3] Gopalan M.A., Vidhyalakshmi S., Sumathi G., Integral points on the Hyperbola,  $x^2 + 6xy + y^2 + 40x + 80y + 40 = 0$  Bessel J Math., 2(3), 2012, 159-164.
- [4] Gopalan M.A., Sumathi G., Vidhyalakshmi S., Observation on the Hyperbola,  $y^2 = 24x^2 + 1$  Bessel J Math, 4, 2013, 21-25.
- [5] Geetha T., Gopalan M.A., Vidhyalakshmi S., Observation on the hyperbola,  $y^2 = 72x^2 + 1$ , Scholars Journal of Physics, Mathematics and Statistics, 1(1), 2014, 1-13.
- [6] Gopalan M.A., Vidhyalakshmi S., Umarani J., Remarkable Observations on the hyperbola  $y^2 = 24x^2 + 1$ , Bulletin of Mathematics and Statistics Research, 1, 2014, 9-12.
- [7] Gopalan M.A., Vidhyalakshmi S., Premalatha E., Vanitha M., Observation on the hyperbola,  $y^2 = 220x^2 + 9$ , Star Research Journal, Vol.4, Issue3(2), 2016, 12-16.
- [8] Ramya S, Kavitha A. On the positive pell equation,  $y^2 = 90x^2 + 31$ , Journal of Mathematics and informatics, 11(1), 2017, 11-17.
- [9] Sumathi G., Observations on the hyperbola,  $y^2 = 150x^2 + 16$ , International Journal of Recent Trends in Engineering and Research, 3(9), 2017, 198-206.
- [10] Meena K, Gopalan M.A., Sivaranjani V, On the positive pell equation  $y^2 = 102x^2 + 33$ . International Journal of Advanced Education and Research 2(1), 2017, 91-96
- [11] Gopalan M.A., Devibala S., Swetha T., On the positive pell equation  $y^2 = 5x^2 + 4$ , IOSR Journal of Mathematics, Vol 13, Issue 5, Ver.1, 2017, 65-69.
- [12] Usha Rani T.R., Dhivya K, On the positive pell equation  $y^2 = 14x^2 + 18$ , International Journal of Academic Research and Development, Volume 3, Issue 3, 2018, 43-50.



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