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On the Positive Pell Equation $y^2 = 32x^2 + 41$

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Abstract: The binary quadratic equation represented by the positive Pellian $y^2 = 32x^2 + 41$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

Keywords: Binary quadratic, hyperbola, integral solutions, parabola, Pell equation. 2010 mathematics subject classification: 11D09

I. INTRODUCTION

A binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, yet another interesting hyperbola given by $y^2 = 32x^2 + 41$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

A. Method of Analysis

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 32x^2 + 41 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = 13$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 32x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{8\sqrt{2}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (17 + 12\sqrt{2})^{n+1} + (17 - 12\sqrt{2})^{n+1}$$

$$g_n = (17 + 12\sqrt{2})^{n+1} - (17 - 12\sqrt{2})^{n+1}, n = -1, 0, 1, 2, \dots$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = f_n + \frac{13}{8\sqrt{2}} g_n$$

$$y_{n+1} = \frac{13}{2} f_n + 8\sqrt{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	2	13
0	73	413
1	2480	14029
2	84247	476573
3	2861918	16189453
4	97220965	549964829

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_{n+1} is alternatively even and odd, y_{n+1} is always odd.

2) Relations among the solutions

a) $x_{n+1} - 17x_{n+2} + 3y_{n+2} = 0$

b) $17x_{n+1} - 577x_{n+2} + 3y_{n+3} = 0$

c) $x_{n+1} - x_{n+3} + 6y_{n+2} = 0$

d) $x_{n+1} - 577x_{n+3} + 102y_{n+3} = 0$

e) $17x_{n+1} - x_{n+2} + 3y_{n+1} = 0$

f) $577x_{n+1} - x_{n+3} + 102y_{n+1} = 0$

g) $96x_{n+1} + 17y_{n+1} - y_{n+2} = 0$

h) $3264x_{n+1} + 577y_{n+1} - y_{n+3} = 0$

i) $96x_{n+1} + 577y_{n+2} - 17y_{n+3} = 0$

j) $1154x_{n+2} - 17x_{n+3} + 3y_{n+3} = 0$

k) $17x_{n+2} - x_{n+3} + 3y_{n+2} = 0$

l) $x_{n+2} - 17x_{n+3} + 3y_{n+3} = 0$

m) $96x_{n+2} - 17y_{n+2} + y_{n+1} = 0$

n) $192x_{n+2} + y_{n+1} - y_{n+3} = 0$

o) $96x_{n+2} - y_{n+3} + 17y_{n+2} = 0$

p) $192x_{n+2} - y_{n+3} + 697y_{n+1} = 0$

q) $96x_{n+3} - 577y_{n+2} + 17y_{n+1} = 0$

r) $3264x_{n+3} - 577y_{n+3} + y_{n+1} = 0$

s) $96x_{n+3} + y_{n+2} - 17y_{n+3} = 0$

t) $y_{n+1} - 102y_{n+2} + y_{n+3} = 0$

3) Each of the following expressions represents a nasty number

$$a) \frac{4}{41}(13x_{2n+3} - 413x_{2n+2} + 123)$$

$$b) \frac{2}{697}(13x_{2n+4} - 14029x_{2n+2} + 4182)$$

$$c) \frac{4}{41}(39y_{2n+2} - 192x_{2n+2} + 123)$$

$$d) \frac{6}{697}(26y_{2n+3} - 4672x_{2n+2} + 1394)$$

$$e) \frac{4}{23657}(39y_{2n+4} - 238080x_{2n+2} + 70971)$$

$$f) \frac{4}{41}(413x_{2n+4} - 14029x_{2n+3} + 123)$$

$$g) \frac{4}{697}(1239y_{2n+2} - 192x_{2n+3} + 2091)$$

$$h) \frac{4}{41}(1239y_{2n+3} - 7008x_{2n+3} + 123)$$

$$i) \frac{4}{697}(1239y_{2n+4} - 238080x_{2n+3} + 2091)$$

$$j) \frac{4}{23657}(42087y_{2n+2} - 192x_{2n+4} + 70971)$$

$$k) \frac{4}{697}(42087y_{2n+3} - 7008x_{2n+4} + 2091)$$

$$l) \frac{4}{41}(42087y_{2n+4} - 238080x_{2n+4} + 123)$$

$$m) \frac{4}{41}(73y_{2n+2} - 2y_{2n+3} + 123)$$

$$n) \frac{4}{697}(1240y_{2n+2} - y_{2n+4} + 2091)$$

$$o) \frac{4}{41}(2480y_{2n+3} - 73y_{2n+4} + 123)$$

4) Each of the following expressions represents a cubical integer

$$a) \frac{1}{123}[78x_{n+2} - 2478x_{n+1} + 26x_{3n+4} - 826x_{3n+3}]$$

$$b) \frac{1}{2091}[39x_{n+3} - 42087x_{n+1} + 13x_{3n+5} - 14029x_{3n+3}]$$

$$c) \frac{1}{41}[78y_{n+1} - 384x_{n+1} + 26y_{3n+3} - 128x_{3n+3}]$$

$$d) \frac{1}{697}[78y_{n+2} - 14016x_{n+1} + 26y_{3n+4} - 4672x_{3n+3}]$$

$$e) \frac{1}{23657} [78y_{n+3} - 476160x_{n+1} + 26y_{3n+5} - 158720x_{3n+3}]$$

$$f) \frac{1}{123} [2478x_{n+3} - 84174x_{n+2} + 826x_{3n+5} - 28058x_{3n+4}]$$

$$g) \frac{1}{697} [2478y_{n+1} - 384x_{n+2} + 826y_{3n+3} - 128x_{3n+4}]$$

$$h) \frac{1}{41} [2478y_{n+2} - 14016x_{n+2} + 826y_{3n+4} - 4672x_{3n+4}]$$

$$i) \frac{1}{697} [2478y_{n+3} - 476160x_{n+2} + 826y_{3n+5} - 158720x_{3n+4}]$$

$$j) \frac{1}{23657} [84174y_{n+1} - 384x_{n+3} + 28058y_{3n+3} - 128x_{3n+5}]$$

$$k) \frac{1}{697} [84174y_{n+2} - 14016x_{n+3} + 28058y_{3n+4} - 4672x_{3n+5}]$$

$$l) \frac{1}{41} [84174y_{n+3} - 476160x_{n+3} + 28058y_{3n+5} - 158720x_{3n+5}]$$

$$m) \frac{1}{123} [438y_{n+1} - 12y_{n+2} + 146y_{3n+3} - 4y_{3n+4}]$$

$$n) \frac{1}{2091} [7440y_{n+1} - 6y_{n+3} + 2480y_{3n+3} - 2y_{3n+5}]$$

$$o) \frac{1}{123} [14880y_{n+2} - 438y_{n+3} + 4960y_{3n+4} - 146y_{3n+5}]$$

5) Each of the following expressions represents a bi-quadratic integer

$$a) \frac{1}{123} [104x_{2n+3} - 3304x_{2n+2} + 26x_{4n+5} - 826x_{4n+4} + 738]$$

$$b) \frac{1}{2091} [52x_{2n+4} - 56116x_{2n+2} + 13x_{4n+6} - 14029x_{4n+4} + 12546]$$

$$c) \frac{1}{41} [104y_{2n+2} - 512x_{2n+2} + 26y_{4n+4} - 128x_{4n+4} + 246]$$

$$d) \frac{1}{697} [104y_{2n+3} - 18688x_{2n+2} + 26y_{4n+5} - 4672x_{4n+4} + 4182]$$

$$e) \frac{1}{23657} [104y_{2n+4} - 634880x_{2n+2} + 26y_{4n+6} - 158720x_{4n+4} + 141942]$$

$$f) \frac{1}{123} [3304x_{2n+4} - 112232x_{2n+3} + 826x_{4n+6} - 28058x_{4n+5} + 738]$$

$$g) \frac{1}{697} [3304y_{2n+2} - 512x_{2n+3} + 826y_{4n+4} - 128x_{4n+5} + 4182]$$

$$h) \frac{1}{41} [3304y_{2n+3} - 18688x_{2n+3} + 826y_{4n+5} - 4672x_{4n+5} + 246]$$

- i) $\frac{1}{697} [3304y_{2n+4} - 634880x_{2n+3} + 826y_{4n+6} - 158720x_{4n+5} + 4182]$
- j) $\frac{1}{23657} [112232y_{2n+2} - 512x_{2n+4} + 28058y_{4n+4} - 128x_{4n+6} + 141942]$
- k) $\frac{1}{697} [112232y_{2n+3} - 18688x_{2n+4} + 28058y_{4n+5} - 4672x_{4n+6} + 4182]$
- l) $\frac{1}{41} [112232y_{2n+4} - 634880x_{2n+4} + 28058y_{4n+6} - 158720x_{4n+6} + 246]$
- m) $\frac{1}{123} [584y_{2n+2} - 16y_{2n+3} + 146y_{4n+4} - 4y_{4n+5} + 738]$
- n) $\frac{1}{2091} [9920y_{2n+2} - 8y_{2n+4} + 2480y_{4n+4} - 2y_{4n+6} + 12546]$
- o) $\frac{1}{123} [19840y_{2n+3} - 584y_{2n+4} + 4960y_{4n+5} - 146y_{4n+6} + 738]$
- b) Each of the following expressions represents a quintic integer
- a) $\frac{1}{123} [260x_{n+2} - 8260x_{n+1} + 130x_{3n+4} - 4130x_{3n+3} + 26x_{5n+6} - 826x_{5n+5}]$
- b) $\frac{1}{2091} [130x_{n+3} - 140290x_{n+1} + 65x_{3n+5} - 70145x_{3n+3} + 13x_{5n+7} - 14029x_{5n+5}]$
- c) $\frac{1}{41} [260y_{n+1} - 1280x_{n+1} + 130y_{3n+3} - 640x_{3n+3} + 26y_{5n+5} - 128x_{5n+5}]$
- d) $\frac{1}{697} [260y_{n+2} - 46720x_{n+1} + 130y_{3n+4} - 23360x_{3n+3} + 26y_{5n+6} - 4672x_{5n+5}]$
- e) $\frac{1}{23657} [260y_{n+3} - 1587200x_{n+1} + 130y_{3n+5} - 793600x_{3n+3} + 26y_{5n+7} - 158720x_{5n+5}]$
- f) $\frac{1}{123} [8260x_{n+3} - 280580x_{n+2} + 4130x_{3n+5} - 140290x_{3n+4} + 826x_{5n+7} - 28058x_{5n+6}]$
- g) $\frac{1}{697} [12390y_{n+1} - 1920x_{n+2} - 128x_{5n+6} + 826y_{5n+5}]$
- h) $\frac{1}{41} [8260y_{n+2} - 46720x_{n+2} + 4130y_{3n+4} - 23360x_{3n+4} + 826y_{5n+6} - 4672x_{5n+6}]$
- i) $\frac{1}{697} [8260y_{n+3} - 1587200x_{n+2} + 4130y_{3n+5} - 793600x_{3n+4} + 826y_{5n+7} - 158720x_{5n+6}]$
- j) $\frac{1}{23657} [280580y_{n+1} - 1280x_{n+3} + 140290y_{3n+3} - 640x_{3n+5} + 28058y_{5n+5} - 128x_{5n+7}]$
- k) $\frac{1}{697} [280580y_{n+2} - 46720x_{n+3} - 23360x_{3n+5} + 140290y_{3n+4} + 28058y_{5n+6} - 4672x_{5n+7}]$
- l) $\frac{1}{41} [280580y_{n+3} - 1587200x_{n+3} + 140290y_{3n+5} - 793600x_{3n+5} + 28058y_{5n+7} - 158720x_{5n+7}]$

$$m) \frac{1}{123} [1460y_{n+1} - 40y_{n+2} + 730y_{3n+3} - 20y_{3n+4} + 146y_{5n+5} - 4y_{5n+6}]$$

$$n) \frac{1}{2091} [24800y_{n+1} - 20y_{n+3} + 12400y_{3n+3} - 10y_{3n+5} + 2480y_{5n+5} - 2y_{5n+7}]$$

$$o) \frac{1}{123} [49600y_{n+2} - 1460y_{n+3} + 24800y_{3n+4} - 730y_{3n+5} + 4960y_{5n+6} - 146y_{5n+7}]$$

B. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

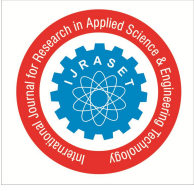
Table: 2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2 - 32Y^2 = 60516$	$(26x_{n+2} - 826x_{n+1}, 146x_{n+1} - 4x_{n+2})$
2	$X^2 - 32Y^2 = 69956496$	$(26x_{n+3} - 28058x_{n+1}, 4960x_{n+1} - 4x_{n+3})$
3	$X^2 - 32Y^2 = 6724$	$(26y_{n+1} - 128x_{n+1}, 26x_{n+1} - 4y_{n+1})$
4	$X^2 - 32Y^2 = 1943236$	$(26y_{n+2} - 4672x_{n+1}, 826x_{n+1} - 4y_{n+1})$
5	$X^2 - 32Y^2 = 2238614596$	$(26y_{n+3} - 158720x_{n+1}, 28058x_{n+1} - 4y_{n+3})$
6	$X^2 - 32Y^2 = 60516$	$(826x_{n+3} - 28058x_{n+2}, 4960x_{n+2} - 146x_{n+3})$
7	$X^2 - 32Y^2 = 1943236$	$(826y_{n+1} - 128x_{n+2}, 26x_{n+1} - 146y_{n+1})$
8	$X^2 - 32Y^2 = 6724$	$(826y_{n+2} - 4672x_{n+2}, 826x_{n+2} - 146y_{n+2})$
9	$X^2 - 32Y^2 = 1943236$	$(826y_{n+3} - 158720x_{n+2}, 28058x_{n+2} - 146y_{n+3})$
10	$X^2 - 32Y^2 = 2238614596$	$(28058y_{n+1} - 128x_{n+3}, 26x_{n+3} - 4960y_{n+1})$
11	$X^2 - 32Y^2 = 1943236$	$(28058y_{n+2} - 4672x_{n+3}, 826x_{n+3} - 4960y_{n+2})$
12	$X^2 - 32Y^2 = 6724$	$(28058y_{n+3} - 158720x_{n+3}, 28058x_{n+3} - 4960y_{n+3})$
13	$32X^2 - Y^2 = 1936512$	$(146y_{n+1} - 4y_{n+2}, 26y_{n+2} - 826y_{n+1})$
14	$32X^2 - Y^2 = 2238607872$	$(4960y_{n+1} - 4y_{n+3}, 26y_{n+3} - 28058y_{n+1})$
15	$32X^2 - Y^2 = 1936512$	$(4960y_{n+2} - 146y_{n+3}, 826y_{n+3} - 28058y_{n+2})$

2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X ,Y)
1	$123X - 32Y^2 = 30258$	$(26x_{2n+3} - 826x_{2n+2}, 146x_{n+1} - 4x_{n+2})$
2	$4182X - 32Y^2 = 34978248$	$(26x_{2n+4} - 28058x_{2n+2}, 4960x_{n+1} - 4x_{n+3})$
3	$41X - 32Y^2 = 3362$	$(26y_{2n+2} - 128x_{2n+2}, 26x_{n+1} - 4y_{n+1})$
4	$697X - 32Y^2 = 971618$	$(26y_{2n+3} - 4672x_{2n+2}, 826x_{n+1} - 4y_{n+2})$
5	$23657X - 32Y^2 = 1119307298$	$(26y_{2n+4} - 158720x_{2n+2}, 28058x_{n+1} - 4y_{n+3})$
6	$123X - 32Y^2 = 30258$	$(826x_{2n+4} - 28058x_{2n+3}, 4960x_{n+2} - 146x_{n+3})$
7	$697X - 32Y^2 = 971618$	$(826y_{2n+2} - 128x_{2n+3}, 26x_{n+2} - 146y_{n+1})$
8	$41X - 32Y^2 = 3362$	$(826y_{2n+3} - 4672x_{2n+3}, 826x_{n+2} - 146y_{n+2})$
9	$697X - 32Y^2 = 971618$	$(826y_{2n+4} - 158720x_{2n+3}, 28058x_{n+2} - 146y_{n+3})$
10	$23657X - 32Y^2 = 1119307298$	$(28058y_{2n+2} - 128x_{2n+4}, 26x_{n+3} - 4960y_{n+1})$
11	$697X - 32Y^2 = 971618$	$(28058y_{2n+3} - 4672x_{2n+4}, 826x_{n+3} - 4960y_{n+2})$
12	$41X - 32Y^2 = 3362$	$(28058y_{2n+4} - 158720x_{2n+4}, 28058x_{n+3} - 4960y_{n+3})$
13	$3936X - Y^2 = 968256$	$(146y_{2n+2} - 4y_{2n+3}, 26y_{n+2} - 826y_{n+1})$
14	$133824X - Y^2 = 1119303936$	$(4960y_{2n+2} - 4y_{2n+4}, 26y_{n+3} - 28058y_{n+1})$
15	$3936X - Y^2 = 968256$	$(4960y_{2n+3} - 146y_{2n+4}, 826y_{n+3} - 28058y_{n+2})$



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