Observation on the Positive Pell Equation

\[ y^2 = 15x^2 + 10 \]

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Abstract: The binary quadratic Diophantine equation represented by the positive Pellian \( y^2 = 15x^2 + 10 \) is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integer solutions

I. INTRODUCTION

The binary quadratic equation of the form \( y^2 = Dx^2 + 1 \) where \( D \) is non-square positive integer, has been selected by various mathematicians for its non-trivial integer solutions when \( D \) takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-16]. In this communication, yet another an interesting equation given by \( y^2 = 15x^2 + 10 \) is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The Positive Pell equation representing hyperbola under consideration is

\[ y^2 = 15x^2 + 10 \]

(1)

The smallest positive integer solutions of (1) are

\[ x_0 = 1, y_0 = 5 \]

To obtain the other solutions of (1), consider the pellian equation

\[ y^2 = 15x^2 + 1 \]

(2)

whose initial solution is given by

\[ x_0 = 1, y_0 = 4 \]

The general solution \( (x_n, y_n) \) of (2) is given by

\[ x_n = \frac{1}{2\sqrt{15}} g_n, \quad y_n = \frac{1}{2} f_n \]

where

\[ f_n = (4 + \sqrt{15})^{n+1} + (4 - \sqrt{15})^{n+1} \]

\[ g_n = (4 + \sqrt{15})^{n} - (4 - \sqrt{15})^{n}, \quad n = -1, 0, 1, \ldots \]

Applying Brahmagupta lemma between \( (x_0, y_0) \) and \( (x_n, y_n) \), the other integer solution of (1) are given by

\[ x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{15}} g_n \]

\[ y_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{15}}{2} g_n \]

The recurrence relations satisfied by the solutions \( x \) and \( y \) are given by

\[ x_{n+3} - 8x_{n+2} + x_{n+1} = 0 \]

\[ y_{n+3} - 8y_{n+2} + y_{n+1} = 0 \]
Some numerical examples of $x_n$ and $y_n$ satisfying (1) are given in the Table 1 below,

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
<td>35</td>
</tr>
<tr>
<td>1</td>
<td>71</td>
<td>275</td>
</tr>
<tr>
<td>2</td>
<td>559</td>
<td>2165</td>
</tr>
<tr>
<td>3</td>
<td>4401</td>
<td>17045</td>
</tr>
<tr>
<td>4</td>
<td>34649</td>
<td>134195</td>
</tr>
</tbody>
</table>

From the above table, we observe some interesting relations among the solutions which are presented below:
Both $x_n$ and $y_n$ values are odd.

A. Relations Among The Solutions Are Given Below

1) $x_{n+3} - 8x_{n+2} + x_{n+1} = 0$
2) $y_{n+3} - 31x_{n+2} + 4x_{n+1} = 0$
3) $8y_{n+1} + 31x_{n+1} - x_{n+3} = 0$
4) $8y_{n+3} - 31x_{n+3} + x_{n+1} = 0$
5) $x_{n+2} - 4x_{n+1} + y_{n+1} = 0$
6) $y_{n+2} - 4y_{n+1} - 15x_{n+1} = 0$
7) $y_{n+3} - 31y_{n+1} - 120x_{n+1} = 0$
8) $4x_{n+2} - x_{n+1} - y_{n+2} = 0$
9) $x_{n+3} - x_{n+1} - 2y_{n+2} = 0$
10) $4y_{n+3} - 31y_{n+2} - 15x_{n+1} = 0$
11) $y_{n+1} + 31x_{n+2} - 4x_{n+3} = 0$
12) $y_{n+2} + 4x_{n+2} - x_{n+3} = 0$
13) $y_{n+3} + x_{n+2} - 4x_{n+3} = 0$
14) $4y_{n+2} - y_{n+1} - 15x_{n+2} = 0$
15) $y_{n+3} - y_{n+1} - 30x_{n+3} = 0$
16) $y_{n+3} - 4y_{n+2} - 15x_{n+2} = 0$
17) $y_{n+3} + 4x_{n+1} - 31x_{n+2} = 0$
18) $4x_{n+3} - x_{n+2} - y_{n+3} = 0$
19) $31y_{n+2} - 4y_{n+1} - 15x_{n+3} = 0$
20) $31y_{n+3} - y_{n+1} - 120x_{n+3} = 0$
21) $4y_{n+3} - y_{n+2} - 15x_{n+3} = 0$
22) $y_{n+3} - 8y_{n+2} + y_{n+1} = 0$
23) $445y_{n+3} + 976y_{n+2} - 31307y_{n+1} = 0$
24) $14640x_{n+3} + 13543y_{n+3} - 952753y_{n+1} = 0$
25) $14640x_{n+2} + 1717y_{n+3} - 120787y_{n+1} = 0$
26) $14640x_{n+1} + 233y_{n+3} - 15743y_{n+1} = 0$
B. Each Of The Following Expressions Represents A Nasty Number

1) \[ 12 + 6x_{2n+3} - 42x_{2n+2} \]
2) \[ \frac{1}{8} [96 + 6x_{2n+4} - 330x_{2n+2}] \]
3) \[ 12 + 6y_{2n+2} - 18x_{2n+2} \]
4) \[ \frac{1}{4} [48 + 6y_{2n+3} - 162x_{2n+2}] \]
5) \[ \frac{1}{31} [372 + 6y_{2n+4} - 1278x_{2n+2}] \]
6) \[ 12 + 42x_{2n+4} - 330x_{2n+3} \]
7) \[ \frac{1}{4} [48 + 42y_{2n+2} - 18x_{2n+3}] \]
8) \[ 12 + 42y_{2n+3} - 162x_{2n+3} \]
9) \[ \frac{1}{4} [48 + 42y_{2n+4} - 1278x_{2n+3}] \]
10) \[ \frac{1}{31} [372 + 330y_{2n+2} - 18x_{2n+4}] \]
11) \[ \frac{1}{4} [48 + 330y_{2n+3} - 62x_{2n+4}] \]
12) \[ 12 + 330y_{2n+4} - 1278x_{2n+4} \]
13) \[ \frac{1}{5} [60 + 54y_{2n+2} - 63y_{2n+3}] \]
14) \[ \frac{1}{40} [480 + 426y_{2n+2} - 6y_{2n+4}] \]
15) \[ \frac{1}{5} [60 + 426y_{2n+3} - 54y_{2n+4}] \]

III. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A CUBICAL INTEGER

1) \[ x_{3n+4} - 7x_{3n+3} + 3x_{n+2} - 21x_{n+1} \]
2) \[ \frac{1}{8} [x_{3n+5} - 55x_{3n+3} + 3x_{n+3} - 165x_{n+1}] \]
3) \[ y_{3n+3} - 3x_{3n+3} - 9x_{n+1} + 3y_{n+1} \]
4) \[ \frac{1}{4} [y_{3n+4} - 27x_{3n+3} - 81x_{n+1} + 3y_{n+2}] \]
5) \[ \frac{1}{31} [y_{3n+5} - 213x_{3n+3} + 3y_{n+3} - 639x_{n+1}] \]
6) \[ [7x_{3n+5} - 55x_{3n+4} + 21x_{n+3} - 165x_{n+2}] \]
7) \[ \frac{1}{4} [7y_{3n+5} - 3x_{3n+4} + 21y_{n+1} - 9x_{n+2}] \]
8) \[ [7y_{3n+4} - 27x_{3n+4} - 81x_{n+2} + 21y_{n+2}] \]
9) \[ \frac{1}{4} [7y_{3n+5} - 213y_{3n+4} - 639y_{n+2} + 21y_{n+3}] \]
10) \[ \frac{1}{31} [55y_{3n+3} - 3x_{3n+5} - 9y_{n+3} + 165y_{n+1}] \]
11) \[ \frac{1}{4} [55y_{3n+4} - 27x_{3n+5} - 81x_{n+3} + 165y_{n+2}] \]
12) \[ [55y_{3n+5} - 213x_{3n+5} - 639x_{n+3} + 165y_{n+3}] \]
13) \( \frac{1}{5} \left[ 9y_{3n+3} - y_{3n+4} + 27y_{n+1} - 3y_{n+2} \right] \)

14) \( \frac{1}{40} \left[ 71y_{3n+3} - y_{3n+5} + 213y_{n+1} - 3y_{n+3} \right] \)

15) \( \frac{1}{5} \left[ 71y_{3n+4} - 9y_{3n+5} + 213y_{n+2} - 27y_{n+3} \right] \)

### IV. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A BIQUADRATIC INTEGER

1) \( x_{4n+5} - 7x_{4n+4} + 28x_{2n+2} + 4x_{2n+3} + 6 \)

2) \( \frac{1}{8} \left[ 55x_{4n+4} - 220x_{2n+2} + 43x_{2n+4} + 48 \right] \)

3) \( \left( x_{4n+4} - 3x_{4n+4} - 12x_{2n+2} + 4y_{2n+2} + 6 \right) \)

4) \( \frac{1}{4} \left[ y_{3n+5} - 27x_{4n+4} + 4y_{2n+3} - 108x_{2n+2} + 24 \right] \)

5) \( \frac{1}{31} \left[ y_{3n+6} - 213y_{4n+4} + 4y_{2n+4} - 852x_{2n+2} + 189 \right] \)

6) \( 7x_{4n+6} - 220x_{2n+3} - 55x_{4n+5} - 28x_{2n+4} + 6 \)

7) \( \frac{1}{4} \left[ 7y_{4n+4} - 3x_{4n+5} - 12x_{2n+3} + 23x_{2n+2} + 29 \right] \)

8) \( \left[ 7y_{4n+5} + 28y_{2n+3} - 27x_{4n+4} - 108x_{2n+3} + 6 \right] \)

9) \( \frac{1}{4} \left[ 7y_{4n+6} + 28y_{2n+4} - 213y_{4n+5} - 852x_{2n+3} + 24 \right] \)

10) \( \frac{1}{31} \left[ 55y_{4n+4} - 3x_{4n+5} - 12x_{2n+4} + 220y_{2n+2} + 189 \right] \)

11) \( \frac{1}{4} \left[ 220y_{2n+3} + 55y_{4n+5} - 27x_{2n+6} - 108x_{2n+4} + 24 \right] \)

12) \( 55y_{4n+6} + 220y_{2n+4} - 213x_{4n+6} - 852x_{2n+4} + 6 \)

13) \( \frac{1}{5} \left[ 9y_{4n+4} - y_{4n+5} + 36y_{2n+2} - 4y_{2n+3} + 30 \right] \)

14) \( \frac{1}{40} \left[ 71y_{4n+4} - y_{4n+6} + 284y_{2n+2} - 4y_{2n+3} + 240 \right] \)

15) \( \frac{1}{5} \left[ 71y_{4n+5} - 9y_{4n+6} + 284y_{2n+3} - 36y_{2n+4} + 30 \right] \)

### V. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A QUINTIC INTEGER

16) \( x_{5n+6} - 7x_{5n+5} + 5x_{3n+5} - 35x_{3n+3} - 70x_{n+1} + 10x_{n+2} \)

17) \( \frac{1}{8} \left[ 55x_{5n+5} + 5x_{3n+5} - 275x_{3n+3} - 550x_{n+1} + 10x_{n+1} \right] \)

18) \( \left[ y_{5n+5} - 3x_{5n+5} + 5y_{3n+5} - 15x_{3n+3} - 30x_{n+1} + 10y_{n+1} \right] \)

19) \( \frac{1}{4} \left[ y_{5n+6} - 27x_{5n+5} + 5x_{3n+4} - 135x_{3n+3} - 270x_{n+1} + 10y_{n+2} \right] \)

20) \( \frac{1}{31} \left[ y_{5n+7} - 213x_{5n+5} - 1065x_{3n+5} + 5y_{3n+5} - 2130x_{n+1} + 10y_{n+3} \right] \)

21) \( \left[ 7y_{5n+7} - 55x_{5n+6} + 35x_{3n+6} - 275x_{3n+4} - 550x_{n+2} + 70x_{n+3} \right] \)

22) \( \frac{1}{4} \left[ 7y_{5n+5} - 3x_{5n+6} - 15x_{3n+4} + 35x_{3n+4} - 30x_{n+2} + 70y_{n+1} \right] \)

23) \( \left[ 7y_{5n+6} - 27x_{5n+6} - 135x_{3n+4} + 35y_{3n+4} - 270x_{n+2} + 70y_{n+2} \right] \)
24) \( \frac{1}{4} \left[ 7y_{n+7} - 213x_{n+6} - 1065x_{n+4} + 35y_{n+5} - 2130x_{n+2} + 70y_{n+3} \right] \)

25) \( \frac{1}{4} \left[ y_{n+6} - 27x_{n+5} + 5x_{n+4} - 135x_{n+3} - 270x_{n+1} + 10y_{n+2} \right] \)

26) \( \frac{1}{3} \left[ 55y_{n+5} - 3x_{n+7} - 15x_{n+5} + 275y_{n+3} - 30x_{n+3} + 550y_{n+1} \right] \)

27) \( \frac{1}{4} \left[ 55y_{n+6} - 27x_{n+7} - 135x_{n+5} + 275y_{n+4} - 270x_{n+3} + 550y_{n+2} \right] \)

28) \( \left[ 55y_{n+7} - 213x_{n+6} - 1065x_{n+5} + 275y_{n+4} - 2130x_{n+3} + 550y_{n+1} \right] \)

29) \( \frac{1}{5} \left[ 9y_{n+5} - y_{n+6} - 5y_{n+4} + 45y_{n+3} + 90y_{n+1} - 10y_{n+2} \right] \)

30) \( \frac{1}{40} \left[ 71y_{n+5} - y_{n+6} + 355y_{n+3} - 5y_{n+4} - 710y_{n+1} - 10y_{n+3} \right] \)

31) \( \frac{1}{5} \left[ 71y_{n+6} - 9y_{n+7} + 355y_{n+5} - 45y_{n+4} + 710y_{n+2} - 90y_{n+3} \right] \)

### VI. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table 2 below:

<table>
<thead>
<tr>
<th>S.NO</th>
<th>Hyperbola</th>
<th>(X, Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 5X^2 - 3Y^2 = 20 )</td>
<td>( (x_{n+2} - 7x_{n+1}, 9x_{n+1} - x_{n+2}) )</td>
</tr>
<tr>
<td>2</td>
<td>( 5X^2 - 3Y^2 = 1280 )</td>
<td>( (x_{n+3} - 55x_{n+1}, 71x_{n+1} - x_{n+3}) )</td>
</tr>
<tr>
<td>3</td>
<td>( 5X^2 - 3Y^2 = 20 )</td>
<td>( (y_{n+1} - 3x_{n+1}, 5x_{n+1} - y_{n+1}) )</td>
</tr>
<tr>
<td>4</td>
<td>( 5X^2 - 3Y^2 = 320 )</td>
<td>( (y_{n+2} - 27x_{n+1}, 35x_{n+1} - y_{n+2}) )</td>
</tr>
<tr>
<td>5</td>
<td>( 5X^2 - 3Y^2 = 19220 )</td>
<td>( (y_{n+3} - 213x_{n+1}, 275x_{n+1} - y_{n+3}) )</td>
</tr>
<tr>
<td>6</td>
<td>( 5X^2 - 3Y^2 = 20 )</td>
<td>( (7x_{n+3} - 55x_{n+2}, 71x_{n+2} - 9x_{n+3}) )</td>
</tr>
<tr>
<td>7</td>
<td>( 5X^2 - 3Y^2 = 320 )</td>
<td>( (7y_{n+1} - 3x_{n+2}, 5x_{n+2} - 9y_{n+1}) )</td>
</tr>
<tr>
<td>8</td>
<td>( 5X^2 - 3Y^2 = 20 )</td>
<td>( (7y_{n+2} - 27x_{n+2}, 35x_{n+2} - 9y_{n+2}) )</td>
</tr>
<tr>
<td>9</td>
<td>( 5X^2 - 3Y^2 = 320 )</td>
<td>( (7y_{n+3} - 213x_{n+2}, 275x_{n+2} - 9y_{n+3}) )</td>
</tr>
<tr>
<td>10</td>
<td>( 5X^2 - 3Y^2 = 19220 )</td>
<td>( (55y_{n+1} - 3x_{n+3}, 5x_{n+3} - 71y_{n+1}) )</td>
</tr>
<tr>
<td>11</td>
<td>( 5X^2 - 3Y^2 = 320 )</td>
<td>( (55y_{n+2} - 27x_{n+3}, 35x_{n+3} - 71y_{n+2}) )</td>
</tr>
<tr>
<td>12</td>
<td>( 5X^2 - 3Y^2 = 20 )</td>
<td>( (55y_{n+3} - 213x_{n+3}, 275x_{n+3} - 71y_{n+3}) )</td>
</tr>
<tr>
<td>13</td>
<td>( 11163X^2 - 20Y^2 = 71443200 )</td>
<td>( (71y_{n+1} - y_{n+3}, 55y_{n+1} - y_{n+3}) )</td>
</tr>
<tr>
<td>14</td>
<td>( 3X^2 - 5Y^2 = 300 )</td>
<td>( (9y_{n+1} - y_{n+2}, y_{n+2} - 7y_{n+1}) )</td>
</tr>
<tr>
<td>15</td>
<td>( 3X^2 - 5Y^2 = 300 )</td>
<td>( (71y_{n+2} - 9y_{n+3}, 7y_{n+3} - 55y_{n+2}) )</td>
</tr>
</tbody>
</table>
2) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table 3 below.

<table>
<thead>
<tr>
<th>S.NO</th>
<th>Parabola</th>
<th>(X,Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(x_{2n+3} - 7x_{2n+2} , 9x_{n+1} - x_{n+3})$</td>
</tr>
<tr>
<td>2</td>
<td>$3Y^2 = 40X - 640$</td>
<td>$(x_{2n+4} - 55x_{2n+2} , 71x_{n+1} - x_{n+3})$</td>
</tr>
<tr>
<td>3</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(y_{2n+2} - 3x_{2n+2} , 5x_{n+1} - y_{n+1})$</td>
</tr>
<tr>
<td>4</td>
<td>$3Y^2 = 20X - 160$</td>
<td>$(y_{2n+3} - 27x_{2n+2} , 35x_{n+1} - y_{n+1})$</td>
</tr>
<tr>
<td>5</td>
<td>$3Y^2 = 155X - 9610$</td>
<td>$(y_{2n+4} - 213x_{2n+2} , 275x_{n+1} - y_{n+1})$</td>
</tr>
<tr>
<td>6</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(7x_{2n+4} - 55x_{2n+3} , 71x_{n+2} - 9x_{n+3})$</td>
</tr>
<tr>
<td>7</td>
<td>$3Y^2 = 20X - 160$</td>
<td>$(7y_{2n+2} - 3x_{2n+2} , 5x_{n+1} - 9y_{n+1})$</td>
</tr>
<tr>
<td>8</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(7y_{2n+3} - 27x_{2n+2} , 35x_{n+2} - 9y_{n+1})$</td>
</tr>
<tr>
<td>9</td>
<td>$3Y^2 = 20X - 160$</td>
<td>$(7y_{2n+4} - 213x_{2n+2} , 275x_{n+2} - 9y_{n+1})$</td>
</tr>
<tr>
<td>10</td>
<td>$3Y^2 = 155X - 9610$</td>
<td>$(55y_{2n+2} - 3x_{2n+4} , 5x_{n+3} - 71y_{n+1})$</td>
</tr>
<tr>
<td>11</td>
<td>$3Y^2 = 20X - 160$</td>
<td>$(55y_{2n+3} - 27x_{2n+4} , 35x_{n+3} - 71y_{n+1})$</td>
</tr>
<tr>
<td>12</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(55y_{2n+4} - 213x_{2n+4} , 275x_{n+3} - 7y_{n+3})$</td>
</tr>
<tr>
<td>13</td>
<td>$3Y^2 = 5X - 10$</td>
<td>$(9y_{2n+2} - y_{2n+3} , y_{n+2} - 7y_{n+1})$</td>
</tr>
<tr>
<td>14</td>
<td>$Y^2 = 22326X - 178608$</td>
<td>$(71y_{2n+2} - y_{2n+4} , 55y_{n+1} - y_{n+3})$</td>
</tr>
<tr>
<td>15</td>
<td>$Y^2 = 3X - 30$</td>
<td>$(71y_{2n+3} - 9y_{2n+4} , 7y_{n+3} - 55y_{n+1})$</td>
</tr>
</tbody>
</table>

### VII. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell equation $y^2 = 15x^2 + 10$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their integer solutions along with suitable properties.

### REFERENCES


