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Observation on the Positive Pell Equation $y^2 = 15x^2 + 10$

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Abstract: The binary quadratic Diophantine equation represented by the positive Pellian $y^2 = 15x^2 + 10$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.
Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integer solutions

I. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer, has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-16]. In this communication, yet another an interesting equation given by $y^2 = 15x^2 + 10$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 15x^2 + 10 \tag{1}$$

The smallest positive integer solutions of (1) are

$$x_0 = 1, y_0 = 5$$

To obtain the other solutions of (1), consider the pellian equation

$$y^2 = 15x^2 + 1 \tag{2}$$

whose initial solution is given by

$$\tilde{x}_0 = 1, \tilde{y}_0 = 4$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{15}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (4+\sqrt{15})^{n+1} + (4-\sqrt{15})^{n+1}$$

$$g_n = (4+\sqrt{15})^{n+1} - (4-\sqrt{15})^{n+1} \quad , n = -1, 0, 1, \ldots$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{15}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{15}}{2} g_n$$

The recurrence relations satisfied by the solutions x and y are given by

$$x_{n+3} - 8x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 8y_{n+2} + y_{n+1} = 0$$

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Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below,

Table 1: Numerical Examples

n	x_n	y_n
0	9	35
1	71	275
2	559	2165
3	4401	17045
4	34649	134195

From the above table, we observe some interesting relations among the solutions which are presented below:
Both x_n and y_n values are odd.

A. Relations Among The Solutions Are Given Below

1) $x_{n+3} - 8x_{n+2} + x_{n+1} = 0$
2) $y_{n+3} - 31x_{n+2} + 4x_{n+1} = 0$
3) $8y_{n+1} + 31x_{n+1} - x_{n+3} = 0$
4) $8y_{n+3} - 31x_{n+3} + x_{n+1} = 0$
5) $x_{n+2} - 4x_{n+1} - y_{n+1} = 0$
6) $y_{n+2} - 4y_{n+1} - 15x_{n+1} = 0$
7) $y_{n+3} - 31y_{n+1} - 120x_{n+1} = 0$
8) $4x_{n+2} - x_{n+1} - y_{n+2} = 0$
9) $x_{n+3} - x_{n+1} - 2y_{n+2} = 0$
10) $4y_{n+3} - 31y_{n+2} - 15x_{n+1} = 0$
11) $y_{n+1} + 31x_{n+2} - 4x_{n+3} = 0$
12) $y_{n+2} + 4x_{n+2} - x_{n+3} = 0$
13) $y_{n+3} + x_{n+2} - 4x_{n+3} = 0$
14) $4y_{n+2} - y_{n+1} - 15x_{n+2} = 0$
15) $y_{n+3} - y_{n+1} - 30x_{n+3} = 0$
16) $y_{n+3} - 4y_{n+2} - 15x_{n+2} = 0$
17) $y_{n+3} + 4x_{n+1} - 31x_{n+2} = 0$
18) $4x_{n+3} - x_{n+2} - y_{n+3} = 0$
19) $31y_{n+2} - 4y_{n+1} - 15x_{n+3} = 0$
20) $31y_{n+3} - y_{n+1} - 120x_{n+3} = 0$
21) $4y_{n+3} - y_{n+2} - 15x_{n+3} = 0$
22) $y_{n+3} - 8y_{n+2} + y_{n+1} = 0$
23) $445y_{n+3} + 976y_{n+2} - 31307y_{n+1} = 0$
24) $14640x_{n+3} + 13543y_{n+3} - 952753y_{n+1} = 0$
25) $14640x_{n+2} + 1717y_{n+3} - 120787y_{n+1} = 0$
26) $14640x_{n+1} + 233y_{n+3} - 15743y_{n+1} = 0$

B. Each Of The Following Expressions Represents A Nasty Number

1) $[12 + 6x_{2n+3} - 42x_{2n+2}]$
2) $\frac{1}{8}[96 + 6x_{2n+4} - 330x_{2n+2}]$
3) $[12 + 6y_{2n+2} - 18x_{2n+2}]$
4) $\frac{1}{4}[48 + 6y_{2n+3} - 162x_{2n+2}]$
5) $\frac{1}{31}[372 + 6y_{2n+4} - 1278x_{2n+2}]$
6) $[12 + 42x_{2n+4} - 330x_{2n+3}]$
7) $\frac{1}{4}[48 + 42y_{2n+2} - 18x_{2n+3}]$
8) $[12 + 42y_{2n+3} - 162x_{2n+3}]$
9) $\frac{1}{4}[48 + 42y_{2n+4} - 1278x_{2n+3}]$
10) $\frac{1}{31}[372 + 330y_{2n+2} - 18x_{2n+4}]$
11) $\frac{1}{4}[48 + 330y_{2n+3} - 162x_{2n+4}]$
12) $[12 + 330y_{2n+4} - 1278x_{2n+4}]$
13) $\frac{1}{5}[60 + 54y_{2n+2} - 6y_{2n+3}]$
14) $\frac{1}{40}[480 + 426y_{2n+2} - 6y_{2n+4}]$
15) $\frac{1}{5}[60 + 426y_{2n+3} - 54y_{2n+4}]$

III. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A CUBICAL INTEGER

1) $[x_{3n+4} - 7x_{3n+3} + 3x_{n+2} - 21x_{n+1}]$
2) $\frac{1}{8}[x_{3n+5} - 55x_{3n+3} + 3x_{n+3} - 165x_{n+1}]$
3) $[y_{3n+3} - 3x_{3n+3} - 9x_{n+1} + 3y_{n+1}]$
4) $\frac{1}{4}[y_{3n+4} - 27x_{3n+3} - 81x_{n+1} + 3y_{n+2}]$
5) $\frac{1}{31}[y_{3n+5} - 213x_{3n+3} + 3y_{n+3} - 639x_{n+1}]$
6) $[7x_{3n+5} - 55x_{3n+4} + 21x_{n+3} - 165x_{n+2}]$
7) $\frac{1}{4}[7y_{3n+3} - 3x_{3n+4} + 21y_{n+1} - 9x_{n+2}]$
8) $[7y_{3n+4} - 27x_{3n+4} - 81x_{n+2} + 21y_{n+2}]$
9) $\frac{1}{4}[7y_{3n+5} - 213y_{3n+4} - 639x_{n+2} + 21y_{n+3}]$
10) $\frac{1}{31}[55y_{3n+3} - 3x_{3n+5} - 9x_{n+3} + 165y_{n+1}]$
11) $\frac{1}{4}[55y_{3n+4} - 27x_{3n+5} - 81x_{n+3} + 165y_{n+2}]$
12) $[55y_{3n+5} - 213x_{3n+5} - 639x_{n+3} + 165y_{n+3}]$

13) $\frac{1}{5}\left[9y_{3n+3} - y_{3n+4} + 27y_{n+1} - 3y_{n+2}\right]$

14) $\frac{1}{40}\left[71y_{3n+3} - y_{3n+5} + 213y_{n+1} - 3y_{n+3}\right]$

15) $\frac{1}{5}\left[71y_{3n+4} - 9y_{3n+5} + 213y_{n+2} - 27y_{n+3}\right]$

IV. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A BIQUADRATIC INTEGER

1) $\left[x_{4n+5} - 7x_{4n+4} + 28x_{2n+2} + 4x_{2n+3} + 6\right]$

2) $\frac{1}{8}\left[x_{4n+6} - 55x_{4n+4} - 220x_{2n+2} + 4x_{2n+4} + 48\right]$

3) $\left[y_{4n+4} - 3x_{4n+4} - 12x_{2n+2} + 4y_{2n+2} + 6\right]$

4) $\frac{1}{4}\left[y_{4n+5} - 27x_{4n+4} + 4y_{2n+3} - 108x_{2n+2} + 24\right]$

5) $\frac{1}{31}\left[y_{4n+6} - 213x_{4n+4} + 4y_{2n+4} - 852x_{2n+2} + 186\right]$

6) $\left[7x_{4n+6} - 220x_{2n+3} - 55x_{4n+5} - 28x_{2n+4} + 6\right]$

7) $\frac{1}{4}\left[7y_{4n+4} - 3x_{4n+5} - 12x_{2n+3} + 28y_{2n+2} + 24\right]$

8) $\left[7y_{4n+5} + 28y_{2n+3} - 27x_{4n+5} - 108x_{2n+3} + 6\right]$

9) $\frac{1}{4}\left[7y_{4n+6} + 28y_{2n+4} - 213x_{4n+5} - 852x_{2n+3} + 24\right]$

10) $\frac{1}{31}\left[55y_{4n+4} - 3x_{4n+6} - 12x_{2n+4} + 220y_{2n+2} + 186\right]$

11) $\frac{1}{4}\left[220y_{2n+3} + 55y_{4n+5} - 27x_{2n+6} - 108x_{2n+4} + 24\right]$

12) $\left[55y_{4n+6} + 220y_{2n+4} - 213x_{4n+6} - 852x_{2n+4} + 6\right]$

13) $\frac{1}{5}\left[9y_{4n+4} - y_{4n+5} + 36y_{2n+2} - 4y_{2n+3} + 30\right]$

14) $\frac{1}{40}\left[71y_{4n+4} - y_{4n+6} + 284y_{2n+2} - 4y_{2n+4} + 240\right]$

15) $\frac{1}{5}\left[71y_{4n+5} - 9x_{4n+6} + 284y_{2n+3} - 36y_{2n+4} + 30\right]$

V. EACH OF THE FOLLOWING EXPRESSIONS REPRESENTS A QUINTIC INTEGER

16) $\left[x_{5n+6} - 7x_{5n+5} + 5x_{3n+5} - 35x_{3n+3} - 70x_{n+1} + 10x_{n+2}\right]$

17) $\frac{1}{8}\left[x_{5n+7} - 55x_{5n+5} + 5x_{3n+5} - 275x_{3n+3} - 550x_{n+1} + 10x_{n+3}\right]$

18) $\left[y_{5n+5} - 3x_{5n+5} + 5y_{3n+3} - 15x_{3n+3} - 30x_{n+1} + 10y_{n+1}\right]$

19) $\frac{1}{4}\left[y_{5n+6} - 27x_{5n+5} + 5x_{3n+4} - 135x_{3n+3} - 270x_{n+1} + 10y_{n+2}\right]$

20) $\frac{1}{31}\left[y_{5n+7} - 213x_{5n+5} - 1065x_{3n+5} + 5y_{3n+5} - 2130x_{n+1} + 10y_{n+3}\right]$

21) $\left[7y_{5n+7} - 55x_{5n+6} + 35x_{3n+5} - 275x_{3n+4} - 550x_{n+2} + 70x_{n+3}\right]$

22) $\frac{1}{4}\left[7y_{5n+5} - 3x_{5n+6} - 15x_{3n+4} + 35x_{3n+4} - 30x_{n+2} + 70y_{n+1}\right]$

23) $\left[7y_{5n+6} - 27x_{5n+6} - 135x_{3n+4} + 35y_{3n+4} - 270x_{n+2} + 70y_{n+2}\right]$

24) $\frac{1}{4}[7y_{5n+7} - 213x_{5n+6} - 1065x_{3n+4} + 35y_{3n+5} - 2130x_{n+2} + 70y_{n+3}]$

25) $\frac{1}{4}[y_{5n+6} - 27x_{5n+5} + 5x_{3n+4} - 135x_{3n+3} - 270x_{n+1} + 10y_{n+2}]$

26) $\frac{1}{31}[55y_{5n+5} - 3x_{5n+7} - 15x_{3n+5} + 275y_{3n+3} - 30x_{n+3} + 550y_{n+1}]$

27) $\frac{1}{4}[55y_{5n+6} - 27x_{5n+7} - 135x_{3n+5} + 275y_{3n+4} - 270x_{n+3} + 550y_{n+2}]$

28) $[55y_{5n+7} - 213x_{5n+7} - 1065x_{3n+5} + 275y_{3n+5} - 2130x_{n+3} + 550y_{n+3}]$

29) $\frac{1}{5}[9y_{5n+5} - y_{5n+6} - 5y_{3n+4} + 45y_{3n+3} + 90y_{n+1} - 10y_{n+2}]$

30) $\frac{1}{40}[71y_{5n+5} - y_{5n+7} + 355y_{3n+3} - 5y_{3n+5} - 710y_{n+1} - 10y_{n+3}]$

31) $\frac{1}{5}[71y_{5n+6} - 9y_{5n+7} + 355y_{3n+4} - 45y_{3n+5} + 710y_{n+2} - 90y_{n+3}]$

VI. REMARKABLE OBSERVATIONS

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table 2 below:

Table 2: Hyperbolas

S.NO	Hyperbola	(X,Y)
1	$5X^2 - 3Y^2 = 20$	$(x_{n+2} - 7x_{n+1}, 9x_{n+1} - x_{n+2})$
2	$5X^2 - 3Y^2 = 1280$	$(x_{n+3} - 55x_{n+1}, 71x_{n+1} - x_{n+3})$
3	$5X^2 - 3Y^2 = 20$	$(y_{n+1} - 3x_{n+1}, 5x_{n+1} - y_{n+1})$
4	$5X^2 - 3Y^2 = 320$	$(y_{n+2} - 27x_{n+1}, 35x_{n+1} - y_{n+2})$
5	$5X^2 - 3Y^2 = 19220$	$(y_{n+3} - 213x_{n+1}, 275x_{n+1} - y_{n+3})$
6	$5X^2 - 3Y^2 = 20$	$(7x_{n+3} - 55x_{n+2}, 71x_{n+2} - 9x_{n+3})$
7	$5X^2 - 3Y^2 = 320$	$(7y_{n+1} - 3x_{n+2}, 5x_{n+2} - 9y_{n+1})$
8	$5X^2 - 3Y^2 = 20$	$(7y_{n+2} - 27x_{n+2}, 35x_{n+2} - 9y_{n+2})$
9	$5X^2 - 3Y^2 = 320$	$(7y_{n+3} - 213x_{n+2}, 275x_{n+2} - 9y_{n+3})$
10	$5X^2 - 3Y^2 = 19220$	$(55y_{n+1} - 3x_{n+3}, 5x_{n+3} - 71y_{n+1})$
11	$5X^2 - 3Y^2 = 320$	$(55y_{n+2} - 27x_{n+3}, 35x_{n+3} - 71y_{n+2})$
12	$5X^2 - 3Y^2 = 20$	$(55y_{n+3} - 213x_{n+3}, 275x_{n+3} - 71y_{n+3})$
13	$11163X^2 - 20Y^2 = 71443200$	$(71y_{n+1} - y_{n+3}, 55y_{n+1} - y_{n+3})$
14	$3X^2 - 5Y^2 = 300$	$(9y_{n+1} - y_{n+2}, y_{n+2} - 7y_{n+1})$
15	$3X^2 - 5Y^2 = 300$	$(71y_{n+2} - 9y_{n+3}, 7y_{n+3} - 55y_{n+2})$

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2) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table 3 below

Table 3: Parabolas

S.NO	Parabola	(X,Y)
1	$3Y^2 = 5X - 10$	$(x_{2n+3} - 7x_{2n+2}\ ,\ 9x_{n+1} - x_{n+3})$
2	$3Y^2 = 40X - 640$	$(x_{2n+4} - 55x_{2n+2}\ ,\ 71x_{n+1} - x_{n+3})$
3	$3Y^2 = 5X - 10$	$(y_{2n+2} - 3x_{2n+2}\ ,\ 5x_{n+1} - y_{n+1})$
4	$3Y^2 = 20X - 160$	$(y_{2n+3} - 27x_{2n+2}\ ,35x_{n+1} - y_{n+2})$
5	$3Y^2 = 155X - 9610$	$(y_{2n+4} - 213x_{2n+2}\ ,\ 275x_{n+1} - y_{n+3})$
6	$3Y^2 = 5X - 10$	$(7x_{2n+4} - 55x_{2n+3}\ ,\ 71x_{n+2} - 9x_{n+3})$
7	$3Y^2 = 20X - 160$	$(7y_{2n+2} - 3x_{2n+3}\ ,\ 5x_{n+2} - 9y_{n+1})$
8	$3Y^2 = 5X - 10$	$(7y_{2n+3} - 27x_{2n+3}\ ,35x_{n+2} - 9y_{n+2})$
9	$3Y^2 = 20X - 160$	$(7y_{2n+4} - 213x_{2n+3}\ ,275x_{n+2} - 9y_{n+3})$
10	$3Y^2 = 155X - 9610$	$(55y_{2n+2} - 3x_{2n+4}\ ,\ 5x_{n+3} - 71y_{n+1})$
11	$3Y^2 = 20X - 160$	$(55y_{2n+3} - 27x_{2n+4}\ ,35x_{n+3} - 71y_{n+2})$
12	$3Y^2 = 5X - 10$	$(55y_{2n+4} - 213x_{2n+4}\ ,275x_{n+3} - 71y_{n+3})$
13	$3Y^2 = 5X - 10$	$(9y_{2n+2} - y_{2n+3}, y_{n+2} - 7y_{n+1})$
14	$\mathbf{Y^2 = 22326X - 1786080}$	$(71y_{2n+2} - y_{2n+4}\ ,\ 55y_{n+1} - y_{n+3})$
15	$Y^2 = 3X - 30$	$(71y_{2n+3} - 9y_{2n+4}\ ,7y_{n+3} - 55y_{n+2})$

VII. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell equation $y^2 = 15x^2 + 10$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their integer solutions along with suitable properties.

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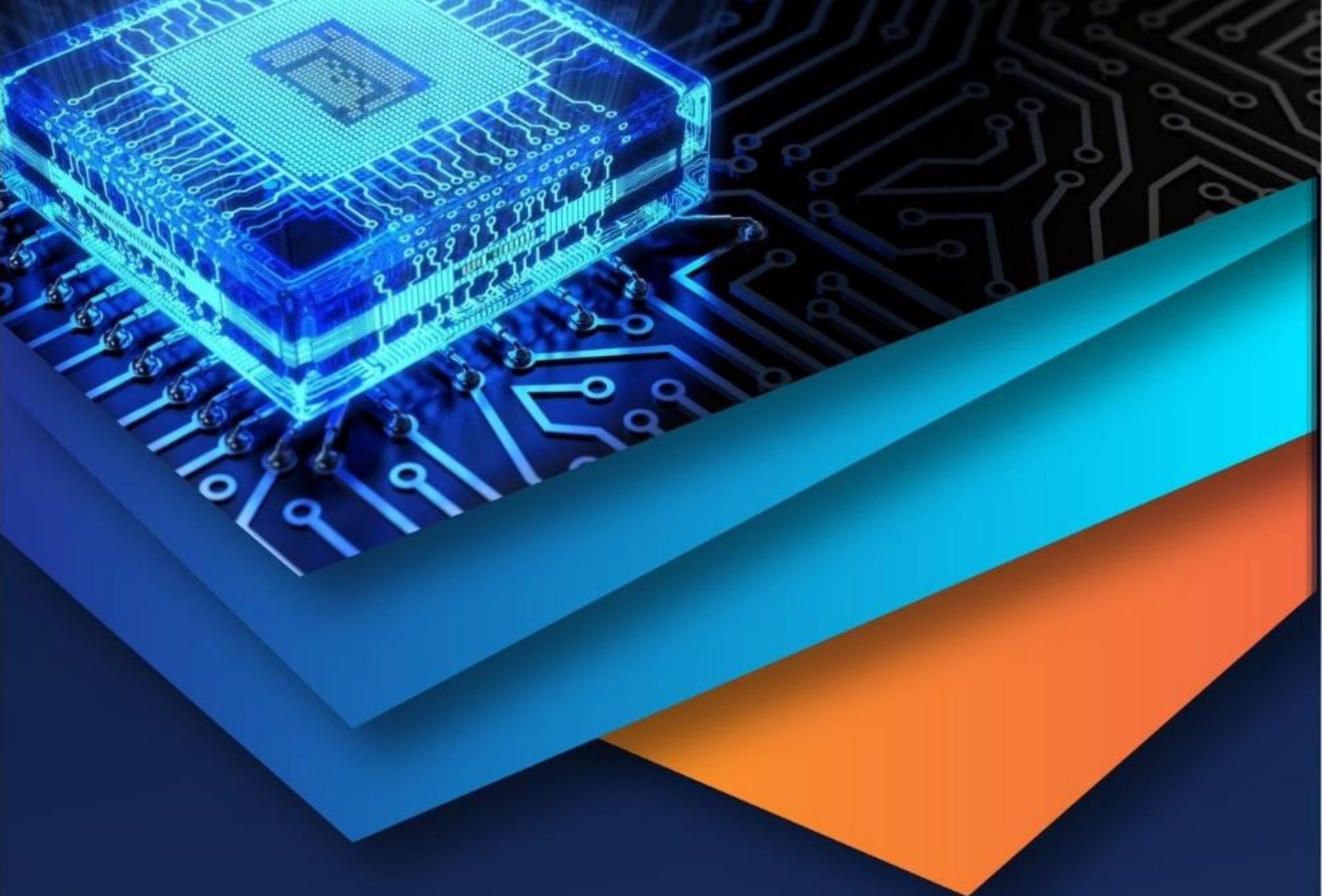

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