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A Scrutiny on Hyperboloid Model and Hypothesis of Hyperbolic Geometry

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Abstract : In this paper we have discussed about the Hyperbolic Geometry which is a branch of Non-Euclidean Geometry. This also includes the Hyperboloid model obtained from paraboloid which includes one sheet and two sheet used in cooling towers. The Hyperbolic Geometry deals with the triangles which has a defect in it whereas in Euclidean Geometry it is zero. The paper includes hypothesis of Hyperbolic Geometry and problems on Hyperboloid model.

Keywords: Hyperbolic Geometry, Hyperboloid model, Defect of triangle, one sheet triangles in Hyperbolic Geometry, cooling towers.

I. INTRODUCTION

A. Hyperbolic Geometry

In mathematics, the Hyperbolic Geometry which is a part of Geometry (is also referred as Bolyai-Lobachevskian geometry or Lobachevskian geometry) is a Non-Euclidean Geometry. The Euclidean Geometry's parallel postulate is replaced with:

For any given line L and point Q not on L, in the plane containing both line L and point Q there are at least two distinct lines through Q that do not intersect L. (compare this with Playfair's axiom, the Euclid's parallel postulate modern version is)

Hyperbolic plane Geometry is also the geometry of a point on a graph which is known as the saddle surface or pseudospherical surfaces, surfaces with a constant negative Gaussian curvature. A contemporary use of Hyperbolic Geometry is in the theory of special relativity, specifically Minkowski spacetime and gyrovector space.

B. Triangles in Hyperbolic Geometry

Unlike Euclidean triangles, where the sum of the angles is always π radians (180^0, a straight angle), in Hyperbolic Geometry the sum of the angles of a Hyperbolic triangle is always strictly less than π radians (180^0, a straight angle). The difference in the triangles is referred to as the defect.

The area of a Hyperbolic triangle is given by its obstruction in radians multiplied by R^2. As a result , all Hyperbolic triangles have an area which is less than or equal to $R^2 \pi$. The area of a Hyperbolic ideal triangle in which all the three angles are 0^0 is equal to this maximum.

As in Euclidean Geometry, each Hyperbolic triangle has a circle inscribed in a triangle . In Hyperbolic Geometry, if all three of its highest point lie on a horocycle or hypercycle, then the triangle has no circumscribed circle. As in Spherical and Elliptical Geometry, in Hyperbolic Geometry if two triangles are same, they must be identical in form.

A Hyperbolic triangle is just three points brought together by (Hyperbolic) two distinct end points. Despite all these resemblances, Hyperbolic triangles are quite different from Euclidean triangles.

In Hyperbolic Geometry, a Hyperbolic triangle is a triangle in the Hyperbolic plane which is completely flat . It consists of three distinct end points called sides or edges and three points called angles or vertices.

A Hyperbolic triangle consists of three points lying in the same straight line and the three segments between them. There are no concept of equal triangles in Hyperbolic Geometry, if two triangles have the same angles then they are congruent.

C. Lines in Hyperbolic Geometry

Single lines in Hyperbolic Geometry have precisely the same properties as single straight lines in Euclidean geometry. For example, two points uniquely define a line, and lines can be extended to an infinite extent.

Two lines which are intersecting have the same properties as two lines which are intersecting in Euclidean Geometry. For example, two lines can intersect in no more than one point, intersecting lines have equal angles which are opposite and intersecting lines of adjacent angles are supplementary.

When we add a third line then there are possessions of intersecting lines that different from lines that are intersecting in Euclidean Geometry. For example, given 2 lines that are intersecting there are infinitely many lines that do not intersect either of the given lines. These possessions all are not dependent of the model used, even if the lines may look fundamentally different.

D. Models of Hyperbolic Geometry
There are four models which are commonly used in Hyperbolic Geometry. These models in Hyperbolic Geometry define plane which is a Hyperbolic and which satisfies the axioms of a Hyperbolic Geometry.
1) The Beltrami-Klein model
2) The Poincar disk model
3) The Poincar half-plane model
4) The Hyperboloid model

E. The Hyperboloid Model
In geometry, a Hyperboloid of revolution, at times called the disc shaped Hyperboloid, is a surface that may be produced by rotating a hyperbola around one of its principal axes. A Hyperboloid is the outside part that may be obtained from a paraboloid of rebellion by deforming it by means of directional scalings, or more generally, of an affine transformation.

A Hyperboloid is a quadric surface characterized by an equation of the second degree that is a surface that may be defined as the zero set of a polynomial of degree two in three variables. Among these surfaces, a Hyperboloid is characterized by not being a cone or a cylinder, having a centre of uniformity, and many planes intersecting into hyperbolas. A Hyperboloid has also three pairwise perpendicular axes of uniformity , and three pairwise perpendicular planes of symmetry.

There are two kinds of Hyperboloids. In the foremost case (+1 in the right-hand side of the equation), one has one-sheet Hyperboloid, which is also called as Hyperbolic Hyperboloid. It is a brought together surface, which has a negative Gaussian curvature at every point. This implies that the tangent plane at any point intersects the Hyperboloid into two lines, and thus that the one-sheet Hyperboloid is a doubly ruled surface.

In the second case (−1 in the right-hand side of the equation), one has a twosheet Hyperboloid, also called elliptic Hyperboloid. The surface has two components which are connected, and a positive Gaussian curvature at every point. Thus the surface is rounded outward in the sense that the tangent plane at every point will intersect the surface only in this point.

1) *Theorem 1: In Hyperbolic Geometry of non-Euclidean case if two triangles are similar they are congruent.*
a) *Note:* This is totally unlike than in the case of Euclidean. (In Euclidean case any three points, when non-collinear, determine a unique triangle and simultaneously a unique plane (i.e., a two-dimensional Euclidean space). It tells us that it is impossible to aggravate or shrink a triangle without distortion.
b) *Proof:* Suppose that the two triangles, triangle PQR and triangle $P^1Q^1R^1$ and these two triangles are similar (ie) they have the same angles but are not congruent (they are not equal). There are also corresponding sides but are not congruent (otherwise they would be congruent using the principle of angle-side-angle). We may assume without loss of generality PQ is greater than P^1Q^1 and PR is greater than P^1R^1.
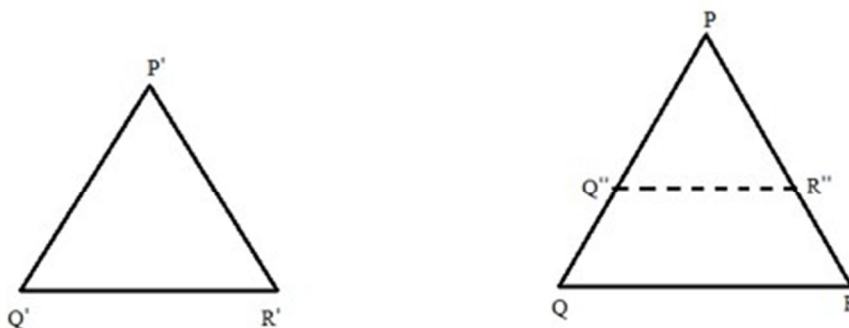
Then by the definition of greater than or equal to there exists a point Q^{11} on the side PQ of the triangle and another point R^{11} on the side PR of the triangle such that $PQ^{11} = P^1Q^1$ and $PR^{11} = P^1R^1$. Then since the angles are same by the postulate (angle-side-angle), then the triangle $P^1Q^1R^1$ is congruent to the triangle $PQ^{11}R^{11}$ (which means the two triangles are equal). Hence $\angle PQ^{11}R^{11} = \angle Q^1$, $\angle PR^{11}Q^{11} = \angle R^1$. But here we also have that $\angle R = \angle R^1$ and $\angle Q = \angle Q^1$, and so we can say $\angle PQ^{11}R^{11} = \angle Q$, and $PR^{11}Q^{11} = \angle R$. Then we can also say by (Alternate Interior Angle theorem) QR and $Q^{11}R^{11}$ cannot intersect. So QR $Q^{11}R^{11}$ forms a quadrilateral which sums to 360^0. This implies that the base lines of the two triangles Q^1R^1 and QR are parallel and hence the quadrilateral $QQ^{11}RR^{11}$ is convex (Not having any interior angles greater than 180^0) and the sum of its angles is exactly 360 degrees, which contradicts the statement below.

In Hyperbolic Geometry, all triangles have angle sum less than 180^0. (From this it follows immediately that all convex quadrilaterals have angle sum less than 360 degrees.)

2) Theorem 2: "Rectangles don't exist in Hyperbolic Geometry"

Using the above statement, we can prove the following Universal Hyperbolic Theorem:

In Hyperbolic Geometry, for every line l and every point P not on l there pass through P at least two distinct parallels through P. Moreover there are many parallels to l through P without any limit.

a) Proof: Draw a perpendicular PQ which meets the base line l at Q and erect a straight line m through P perpendicular to PQ like in the figure below.
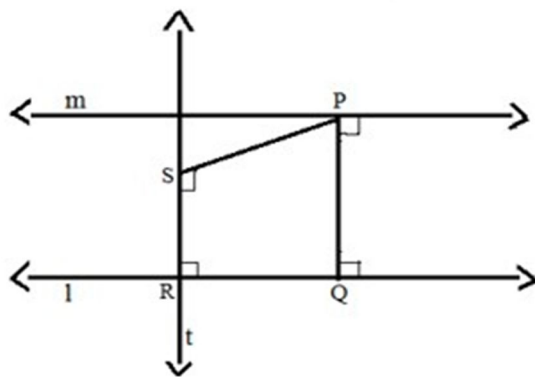
Let us consider another point R on the line l, erect perpendicular t to l through R and draw the perpendicular PS to t. Now, let S be the foot of the perpendicular to l through P. Now the line PS is parallel to the line l, since both PS and l are perpendicular to t.

Assume that m and PS are now on the same line (so S belongs to m) and at the same time PS is not equal to m. This would mean that PQRS is a Rectangle, which contradicts the statement ("Rectangles don't exist in Hyperbolic Geometry"). Hence there are two distinct parallels to l through P. By changing over R, we get infinitely many parallels.

Note: If one Rectangle exists all triangles have defect 0.

F. Defect Of A Triangle

The Defect of a triangle ABC is the number of the form defect (Δ

$$ABC) = 180^0 \quad (\angle A + \angle B + \angle C)$$

In Euclidean Geometry we are customary to having triangles whose defect is zero. The Saccheri - Legendre Theorem indicates that it may not be so. If we have one triangle which is defective then all of the sub and super-triangles are defective.
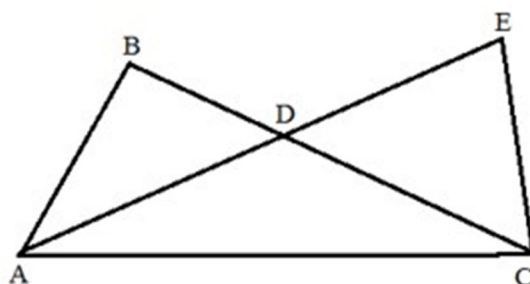
By defective we mean that the triangles have positive defect.

1) Theorem 3: The sum of the degree measure of the three angles in any triangle is less than or equal to 180degrees.

$$\angle A + \angle B + \angle C \le 180^\circ$$

a) Proof: Let us assume the contrary i.e., assume that we have the triangle ABC where $A+B+C>180^\circ$.

So there is an x ϵR^+ so that

$\angle A + \angle B + \angle C = 180^\circ + x$

Let D be the midpoint of BC and let E be the unique point on the ray AD so that DE is congruent to AD. Then by SAS Δ BAD is congruent to ΔCED. This makes $\angle B = \angle ECD$, $\angle E = \angle BAD$

Thus

$$\angle A + \angle B + \angle C = (\angle DAB + \angle CAE) + \angle B + \angle ACB$$
$$= \angle E + \angle EAC + (\angle DCE + \angle ACD)$$
$$= \angle E + \angle A + \angle C$$

So, ΔABC and ΔACE have the same angle sum even though they need not be congruent. Note that $\angle BAE + \angle CAE = \angle BAC$, hence

$$\angle CEA + \angle CAE = \angle BAC$$

It is impossible for both of the angles $\angle CEA$ and $\angle CAE$ to have angle measure greater than $\frac{1}{2} \angle BAC$, so that atleast one of the angles has angle measure less than or equal to $\frac{1}{2} \angle BAC$. Therefore there is triangle ACE so that the angle sum is $180^\circ + x$ but in which one angle has measure less than or equal to $\frac{1}{2} \angle A$. Repeat this construction to get another triangle with angle sum $180^\circ + x$ but in which one angle has measure less than or equal to $\frac{1}{4} \angle A$. Now there is an $n \epsilon z^+$ so that

$$\frac{1}{2^n} \angle A \leq x,$$

By the Archimedean property of real numbers. Thus, after a finite number of iterations of the above construction we obtain a triangle with angle sum $180 + x$ in which one angle has measure less than or equal to

$$\frac{1}{2^n} \angle A \leq x,$$

Then the other two angles must sum to a number greater than 180° contradicting the Corollary

G. Corollary

In triangle ABC the sum of the degree measure of two angles is less than or equal to the degree measure of their remote exterior angle.

1) *Theorem 4:* Let Δ ABC be a triangle and let D be a point between A and B. Then

$$\text{Defect } (\Delta ABC) = \text{defect } (\Delta ACD) + \text{defect } (\Delta BCD)$$

a) Proof

Since the ray DC lies in $\angle ACB$, we know that

$\angle ACB = \angle ACD + \angle BCD$,

And since $\angle ADC$ and $\angle BDC$ are supplementary angles $\angle ADC + \angle BDC = 180^0$

Therefore defect $(\Delta ABC) = 180^0 - (\angle A + \angle B + \angle C)$
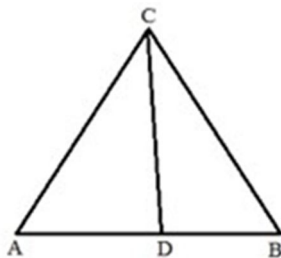
$= 180^0 - (\angle A + \angle B + \angle ACD + \angle BCD)$

$180^0 + 180^0 - (\angle A + \angle B + \angle ACD + \angle BCD + \angle ADC + \angle BDC)$

Defect $(\Delta ABC) = $ defect $(\Delta ACD) + $ defect (ΔBCD)

Defect $(\Delta ABC) = $ defect $(\Delta ACD) + $ defect (ΔBCD)

Corollary: defect $(\Delta ABC) = 0$ if and only if defect $(\Delta ACD) = $ defect $(\Delta BCD) = 0$

2) *Problem 1:* In the setting of Hyperbolic Geometry consider the triangle Δ ABC as shown in the figure. Suppose that m∠1 = 12°, m ∠2 = 25^0, m∠ 3 = 80^0, m∠ 4 = 100^0, and defect (Δ ABC) = 118. Find m∠ ACB.

3) *Sol:* We have 118^0 = d (Δ ABC) = 180^0 - m∠ 1 - m∠ 2 – m ∠ACB

= 180^0 - 12^0 - 25^0 - m∠ ACB

= 143^0 - m ∠ACB

Therefore m ∠ACB = 143^0 - 118^0 = 25^0.
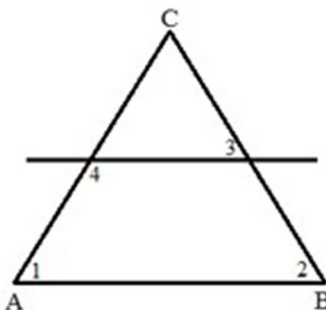
4) *Problem 2:* A cooling tower for a nuclear reactor is to be constructed in the shape of the Hyperboloid of one sheet. The base caliber is 250m and the minimum caliber, 500m above the base is 200m. Find an equation describing the shape of the tower in the co-ordinates where the origin is at the centre of the narrowest path of the tower. In particular use co-ordinates at the origin 500m above the ground. (Assume the centre is at the origin with axis z-axis and the minimum diameter is at the centre).

H. Solution

We are obviously meant to assume that the horizontal cross sections are circles, so in our equation we can take a=b. Also we are given that the minimum diameter is 200, so provided we take the origin at that level (which they recommend), we will have $a = b = 100$ (because then the four points (±100,±100,0) will lie on the surface. We are given that the cross section at $z = -500$ is a circle diameter 125, so for example, the point (125,0,−500) lies on the surface. That is enough to find c.

If we position the hyperboloid on co-ordinate axes so that it is centred at the origin with axis the z-axis then its equation is given by

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=1$$

Horizontal traces in z=k are

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 = 1+\left(\frac{k}{c}\right)^2$$

a family of ellipses, but we know that the marks are circles so that we must have a=b.

The trace in z=0 is

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 =1$$

Or $x^2 +y^2=a^2$ and since the maximum radius of 100m occurs there, we must have a=100. The base tower base is the trace in z= -500 given by

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ but a=100 so the trace is

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \frac{k^2}{c^2}$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1 + \frac{k^2}{c^2}$

$x^2 + y^2 = a^2 \left(1 + \frac{k^2}{c^2}\right)$

$x^2 + y^2 = a^2 + \frac{a^2 k^2}{c^2}$

$x^2 + y^2 = 100^2 + \frac{100^2 (500)^2}{c^2}$

$x^2 + y^2 = 100^2 + \frac{50000^2}{c^2}$

We know the base of the circle of radius 140, so we have

$100^2 + \frac{50000^2}{c^2} = 125^2$

$c^2 = \frac{50000^2}{125^2 - 100^2}$

$c^2 = \frac{50000 * 50000}{5625}$

$c^2 = \frac{4000000}{9}$

And an equation for the tower is

$\left(\frac{x}{100}\right)^2 + \left(\frac{y}{100}\right)^2 - \frac{9z^2}{4000000} = 1$

II.   CONCLUSION

A branch of Non - Euclidean Geometry is the Hyperbolic Geometry. In Hyperbolic Geometry all the three angles in a triangle will sum to a number less than 180 degrees. The defect of the triangle is known as the sum of the three angles subtracted from 180 degrees. In Hyperbolic Geometry if two triangles are similar they are congruent and a lemma states that "Rectangles don't exist in Hyperbolic Geometry". One of the model of Hyperbolic Geometry is the Hyperboloid model. Its main application is the cooling towers which is used to reduce the heat and cool the fluid in any reactor. A reactor is constructed in the shape of one sheet of Hyperboloid in which the diameter is given and the centre is along the z-axis which is used to find the equation of the tower.

REFERENCES

[1]   https://en.m.wikipedia.org
[2]   www.math.cornell.edu
[3]   https://brilliant.org
[4]   www.chegg.com
[5]   https://www.reddit.com
[6]   https://www.jastor.org
[7]   https://www.d.umn.edu
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