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On the Positive Pell Equation $y^2 = 23x^2 + 13$

A. Kavitha¹, K. Maragathavalli²

¹Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil nadu, India.

²Mphil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil nadu, India.

Abstract: The binary quadratic Diophantine equation represented by the positive Pellian $y^2 = 23x^2 + 13$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 23x^2 + 13$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 23x^2 + 13 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 6 \quad D = 23$$

The Pellian equation is

$$y^2 = 23x^2 + 1 \tag{2}$$

The initial solution of Pellian equation is

$$\tilde{x}_0 = 5, \tilde{y}_0 = 24,$$

The general solution (x_n, y_n) of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{23}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (24 + 5\sqrt{23})^{n+1} + (24 - 5\sqrt{23})^{n+1}$$

$$g_n = (24 + 5\sqrt{23})^{n+1} - (24 - 5\sqrt{23})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2} f_n + \frac{6}{2\sqrt{23}} g_n$$

$$y_{n+1} = \frac{6}{2} f_n + \frac{23}{2\sqrt{23}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 48x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 48y_{n+2} + y_{n+1} = 0 \quad n=0,1,2,3,\dots$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	1	6
1	54	259
2	2591	12426
3	124314	596189
4	5964481	28604646

From the above table, we observe some interesting relations among the solutions which are presented below.

- A. x_n values are odd and even alternatively.
- B. y_n values are even and odd alternatively.
- C. Each of the following expression is a nasty number:

- 1) $\frac{6}{13} [26 + 12y_{2n+2} - 46x_{2n+2}]$
- 2) $\frac{6}{65} [130 + 12x_{2n+3} - 518x_{2n+2}]$
- 3) $\frac{6}{1560} [3120 + 6x_{2n+4} - 12426x_{2n+2}]$
- 4) $\frac{6}{312} [624 + 12y_{2n+3} - 2484x_{2n+2}]$
- 5) $\frac{6}{14963} [29926 + 12y_{2n+4} - 119186x_{2n+2}]$
- 6) $\frac{6}{312} [624 + 518y_{2n+2} - 46x_{2n+3}]$
- 7) $\frac{6}{14963} [29926 + 24852y_{2n+2} - 46x_{2n+4}]$
- 8) $\frac{6}{1495} [2990 + 2484y_{2n+2} - 46y_{2n+3}]$
- 9) $\frac{6}{71760} [143520 + 119186y_{2n+2} - 46y_{2n+4}]$
- 10) $\frac{6}{13} [26 + 518y_{2n+3} - 2484x_{2n+3}]$
- 11) $\frac{6}{65} [130 + 518x_{2n+4} - 24852x_{2n+3}]$
- 12) $\frac{6}{312} [624 + 518y_{2n+4} - 119186x_{2n+3}]$
- 13) $\frac{6}{156} [312 + 12426y_{2n+3} - 1242x_{2n+4}]$

$$14) \frac{6}{13} [26 + 24852y_{2n+4} - 119186x_{2n+4}]$$

$$15) \frac{6}{1495} [2990 + 119186y_{2n+3} - 2484y_{2n+4}]$$

D. Each Of The Following Expressions Is A Cubical Integer

$$1) \frac{1}{13} [12y_{3n+3} - 46x_{3n+3} + 36y_{n+1} - 138x_{n+1}]$$

$$2) \frac{1}{65} [12x_{3n+4} - 518x_{3n+3} + 36x_{n+2} - 1554x_{n+1}]$$

$$3) \frac{1}{1560} [6x_{3n+5} - 12426x_{3n+3} + 18x_{n+3} - 37278x_{n+1}]$$

$$4) \frac{1}{132} [12y_{3n+4} - 2484x_{3n+3} + 36y_{n+2} - 7452x_{n+1}]$$

$$5) \frac{1}{14963} [12y_{3n+5} - 119186x_{3n+3} + 36y_{n+3} - 357558x_{n+1}]$$

$$6) \frac{1}{312} [518y_{3n+3} - 46x_{3n+4} + 1554y_{n+1} - 138x_{n+2}]$$

$$7) \frac{1}{14963} [24852y_{3n+3} - 46x_{3n+5} + 149112y_{n+1} - 138x_{n+3}]$$

$$8) \frac{1}{1495} [2484y_{3n+3} - 46y_{3n+4} + 7452y_{n+1} - 138y_{n+2}]$$

$$9) \frac{1}{71760} [119186y_{3n+3} - 46y_{3n+5} + 357558y_{n+1} - 138y_{n+3}]$$

$$10) \frac{1}{65} [518x_{3n+5} - 24852x_{3n+4} + 1554x_{n+3} - 74556x_{n+2}]$$

$$11) \frac{1}{13} [518y_{3n+4} - 2484x_{3n+4} + 1554y_{n+2} - 7452x_{n+2}]$$

$$12) \frac{1}{312} [518y_{3n+5} - 119186x_{3n+4} + 1554y_{n+3} - 357558x_{n+2}]$$

$$13) \frac{1}{156} [12426y_{3n+4} - 1242x_{3n+5} + 37278y_{n+2} - 3726x_{n+3}]$$

$$14) \frac{1}{13} [24852y_{3n+5} - 119186x_{3n+5} + 74556y_{n+3} - 357558x_{n+3}]$$

$$15) \frac{1}{1495} [119186y_{3n+4} - 2484y_{3n+5} + 357558y_{n+2} - 7452y_{n+3}]$$

E. Each Of The Following Expressions Is A Biquadratic Integer

$$1) \frac{1}{13} [12y_{4n+4} - 46x_{4n+4} + 48y_{2n+2} - 184x_{2n+2} + 78]$$

- 2) $\frac{1}{65}[12x_{4n+5} - 518x_{4n+4} + 48x_{2n+3} - 2072x_{2n+2} + 390]$
- 3) $\frac{1}{1560}[6x_{4n+6} - 12426x_{4n+4} + 24x_{2n+4} - 49704x_{2n+2} + 9360]$
- 4) $\frac{1}{1312}[12y_{4n+5} - 2484x_{4n+4} + 48y_{2n+3} - 9936x_{2n+2} + 1872]$
- 5) $\frac{1}{14963}[12y_{4n+6} - 119186x_{4n+4} + 48y_{2n+4} - 476744x_{2n+2} + 89778]$
- 6) $\frac{1}{312}[518y_{4n+4} - 46x_{4n+5} + 2708y_{2n+2} - 184x_{2n+3} + 1872]$
- 7) $\frac{1}{14963}[24852y_{4n+4} - 46x_{4n+6} + 99408y_{2n+2} - 184x_{2n+4} + 89778]$
- 8) $\frac{1}{1495}[2484y_{4n+4} - 46y_{4n+5} + 9936y_{2n+2} - 184y_{2n+3} + 8970]$
- 9) $\frac{1}{71760}[11986y_{4n+4} - 46y_{4n+6} + 47664y_{2n+2} - 184y_{2n+4} + 430560]$
- 10) $\frac{1}{65}[518x_{4n+6} - 24852x_{4n+5} + 2072x_{2n+4} - 99408x_{2n+3} + 390]$
- 11) $\frac{1}{13}[518y_{4n+5} - 2484x_{4n+5} + 2072y_{2n+3} - 9836x_{2n+3} + 78]$
- 12) $\frac{1}{312}[518y_{4n+6} - 119186x_{4n+5} + 2072y_{2n+4} - 476744x_{2n+3} + 1872]$
- 13) $\frac{1}{156}[12426y_{4n+5} - 1242x_{4n+6} + 49704y_{2n+3} - 4968x_{2n+4} + 936]$
- 14) $\frac{1}{13}[24852y_{4n+6} - 119186x_{4n+6} + 99408y_{2n+4} - 476744x_{2n+4} + 78]$
- 15) $\frac{1}{1495}[119186y_{4n+5} - 2484y_{4n+6} + 476744y_{2n+3} - 9936y_{2n+4} + 8970]$

F. Each Of The Following Expression Is A Quintic Integer

- 1) $\frac{1}{13}[12y_{5n+5} - 46x_{5n+5} + 60y_{3n+3} - 230x_{3n+3} + 120y_{n+1} - 460x_{n+1}]$
- 2) $\frac{1}{65}[12x_{5n+6} - 518x_{5n+5} + 60x_{3n+4} - 2590x_{3n+3} + 120x_{n+2} + 1036x_{n+1}]$
- 3) $\frac{1}{1560}[6x_{5n+7} - 12426x_{5n+5} - 30x_{3n+5} - 62130x_{3n+3} + 60x_{n+3} - 124260x_{n+1}]$
- 4) $\frac{1}{312}[12y_{5n+6} - 2484x_{5n+5} + 60y_{3n+4} - 12420x_{3n+3} + 120y_{n+2} - 24840x_{n+1}]$
- 5) $\frac{1}{14963}[12y_{5n+7} - 119186x_{5n+5} + 60y_{3n+5} - 595930x_{3n+3} + 120y_{n+3} - 1191860x_{n+1}]$

- 6) $\frac{1}{312} [518y_{5n+5} - 46x_{5n+6} + 2590y_{3n+3} - 230x_{3n+4} + 5180y_{n+1} - 460x_{n+2}]$
- 7) $\frac{1}{14963} [24852y_{5n+5} - 46x_{5n+7} + 124260y_{3n+3} - 230x_{3n+5} + 248520y_{n+1} - 460x_{n+3}]$
- 8) $\frac{1}{1495} [2484y_{5n+5} - 46y_{5n+7} + 12420y_{3n+3} - 230y_{3n+5} + 24840y_{n+1} - 460y_{n+2}]$
- 9) $\frac{1}{71760} [119186y_{5n+5} - 46y_{5n+7} + 595930y_{3n+3} - 230y_{3n+5} + 1191860y_{n+1} - 460y_{n+3}]$
- 10) $\frac{1}{65} [518x_{5n+7} - 24852x_{5n+6} + 2590x_{3n+5} - 124260x_{3n+4} + 5180x_{n+3} - 248520x_{n+2}]$
- 11) $\frac{1}{13} [518y_{5n+6} - 2484x_{5n+6} + 2590y_{3n+4} - 12420x_{3n+4} + 5180y_{n+2} - 24840x_{n+2}]$
- 12) $\frac{1}{312} [518y_{5n+7} - 119186x_{5n+6} + 2990y_{3n+5} - 595930x_{3n+4} + 5180y_{n+3} - 1191860x_{n+2}]$
- 13) $\frac{1}{156} [12426y_{5n+6} - 1242x_{5n+7} + 62130y_{3n+4} - 6210x_{3n+5} + 124260y_{n+2} - 12420x_{n+3}]$
- 14) $\frac{1}{13} [24852y_{5n+7} - 119186x_{5n+7} + 124260y_{3n+5} - 595930x_{3n+5} + 248520y_{n+3} - 1191860x_{n+3}]$
- 15) $\frac{1}{1495} [119186y_{5n+6} - 2484y_{5n+7} + 595930y_{3n+4} - 12420y_{3n+5} + 1191860y_{n+2} - 24840y_{n+3}]$

G. Relations Among The Solutions Are Given Below

- 1) $x_{n+2} = 5y_{n+1} + 24x_{n+1}$
- 2) $x_{n+3} = 240y_{n+1} + 1151x_{n+1}$
- 3) $y_{n+2} = 24y_{n+1} + 115x_{n+1}$
- 4) $y_{n+3} = 1151y_{n+1} + 5520x_{n+1}$
- 5) $x_{n+3} = 48x_{n+2} - x_{n+1}$
- 6) $5y_{n+2} = 24x_{n+2} - x_{n+1}$
- 7) $5y_{n+3} = 1151x_{n+2} - 24x_{n+1}$
- 8) $10y_{n+2} = x_{n+3} - x_{n+1}$
- 9) $240y_{n+3} = 1151x_{n+3} - x_{n+1}$
- 10) $24y_{n+3} = 1151y_{n+2} + 115x_{n+1}$
- 11) $24x_{n+3} = 5y_{n+1} + 1151x_{n+2}$
- 12) $24y_{n+2} = y_{n+1} + 115x_{n+2}$
- 13) $24y_{n+3} = 24y_{n+1} + 5520x_{n+2}$
- 14) $1151y_{n+2} = 24y_{n+1} + 115x_{n+3}$
- 15) $1151y_{n+3} = y_{n+1} + 5520x_{n+3}$

$$16) 115y_{n+3} = 5520y_{n+2} - 115y_{n+1}$$

$$17) 5y_{n+2} = x_{n+3} - 24x_{n+2}$$

$$18) 5y_{n+3} = 24x_{n+3} - x_{n+2}$$

$$19) y_{n+3} = 24y_{n+2} + 115x_{n+2}$$

$$20) 24y_{n+3} = y_{n+2} + 115x_{n+3}$$

III.REMARKABLE OBSERVATION

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 23X^2 = 676$	$(12x_{n+1} - 2y_{n+1}, 12y_{n+1} - 46x_{n+1})$
2	$Y^2 - 23X^2 = 169000$	$(108x_{n+1} - 2x_{n+2}, 12x_{n+2} - 518x_{n+1})$
3	$Y^2 - 23X^2 = 9734400$	$(2591x_{n+1} - x_{n+3}, 6x_{n+3} - 12426x_{n+1})$
4	$Y^2 - 23X^2 = 3896376$	$(518x_{n+1} - 2y_{n+2}, 12y_{n+2} - 2484x_{n+1})$
5	$Y^2 - 23X^2 = 895565476$	$(24852x_{n+1} - 2y_{n+3}, 12y_{n+3} - 119186x_{n+1})$
6	$Y^2 - 23X^2 = 389376$	$(12x_{n+2} - 108y_{n+1}, 518y_{n+1} - 46x_{n+2})$
7	$Y^2 - 23X^2 = 895565476$	$(12x_{n+3} - 5182y_{n+1}, 24852y_{n+1} - 46x_{n+3})$
8	$Y^2 - 23X^2 = 8940100$	$(12y_{n+2} - 518y_{n+1}, 2484y_{n+1} - 46y_{n+2})$
9	$Y^2 - 23X^2 = 5149497600$	$(6y_{n+3} - 12426y_{n+1}, 59593y_{n+1} - 23y_{n+3})$
10	$Y^2 - 23X^2 = 16900$	$(5182x_{n+2} - 24852x_{n+3}, 518x_{n+3} - 24852x_{n+2})$
11	$Y^2 - 23X^2 = 676$	$(518x_{n+2} - 108y_{n+2}, 518y_{n+2} - 2484x_{n+2})$
12	$Y^2 - 23X^2 = 389376$	$(24852x_{n+2} - 108y_{n+3}, 518y_{n+3} - 119186x_{n+2})$
13	$Y^2 - 23X^2 = 97344$	$(259x_{n+3} - 2591y_{n+2}, 12426y_{n+2} - 1242x_{n+3})$
14	$Y^2 - 23X^2 = 676$	$(24852x_{n+3} - 5182y_{n+3}, 24852y_{n+3} - 119186x_{n+3})$
15	$Y^2 - 23X^2 = 8940100$	$(518y_{n+3} - 24852y_{n+2}, 119186y_{n+2} - 2484y_{n+3})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X, Y)
1	$13Y - 23X^2 = 676$	$(12x_{n+1} - 2y_{n+1}, 12y_{2n+2} - 46x_{2n+2} + 26)$
2	$65Y - 23X^2 = 16900$	$(108x_{n+1} - 2x_{n+2}, 12x_{2n+3} - 518x_{2n+2} + 130)$
3	$1560Y - 23X^2 = 9734400$	$(2591x_{n+1} - x_{n+3}, 6x_{2n+4} - 12426x_{2n+2} + 3120)$
4	$312Y - 23X^2 = 389376$	$(518x_{n+1} - 2y_{n+2}, 12y_{2n+3} - 2484x_{2n+2} + 624)$
5	$14963Y - 23X^2 = 895565476$	$(24852x_{n+1} - 2y_{n+3}, 12y_{2n+4} - 119186x_{2n+2} + 29926)$
6	$312Y - 23X^2 = 389376$	$(12x_{n+2} - 108y_{n+1}, 518y_{2n+2} - 46x_{2n+3} + 624)$
7	$14963Y - 23X^2 = 895565476$	$(12x_{n+3} - 5182y_{n+1}, 24852y_{2n+2} - 46x_{2n+4} + 29926)$
8	$1495Y - 23X^2 = 8940100$	$(12y_{n+2} - 518y_{n+1}, 2484y_{2n+2} - 46y_{2n+3} + 2990)$
9	$35880Y - 23X^2 = 5149497600$	$(6y_{n+3} - 12426y_{n+1}, 59593y_{2n+2} - 23y_{2n+4} + 71760)$
10	$65Y - 23X^2 = 16900$	$(5182x_{n+2} - 108x_{n+3}, 518x_{2n+4} - 24852x_{2n+3} + 130)$
11	$13Y - 23X^2 = 676$	$(518x_{n+2} - 108y_{n+2}, 518y_{2n+3} - 2484x_{2n+3} + 26)$
12	$312Y - 23X^2 = 389376$	$(24852x_{n+2} - 108y_{n+3}, 518y_{2n+4} - 119186x_{2n+3} + 624)$
13	$156Y - 23X^2 = 97344$	$(259x_{n+3} - 2591y_{n+2}, 12426y_{2n+3} - 1242x_{2n+4} + 312)$
14	$13Y - 23X^2 = 676$	$(24852x_{n+3} - 5182y_{n+3}, 24852y_{2n+4} - 119186x_{2n+4} + 26)$
15	$1495Y - 23X^2 = 8940100$	$(518y_{n+3} - 24852y_{n+2}, 119186y_{2n+3} - 2484y_{2n+4} + 2990)$

C. Some Special Cases Among The Solutions Are Given Below

- 1) $P_y^{10}(t_{3,x+1})^2 = 207P_x^6(t_{3,y})^2 + 13(t_{3,y})^2(t_{3,x+1})^2$
- 2) $9P_y^6(t_{3,x})^2 = 23P_x^{10}(t_{3,y+1})^2 + 13(t_{3,x})^2(t_{3,y+1})^2$
- 3) $P_y^{10}(t_{3,2x-2})^2 = 23(6P_{x-1}^4)^2(t_{3,y})^2 + 13(t_{3,y})^2(t_{3,2x-2})^2$
- 4) $36P_{y-1}^8(t_{3,x})^2 = 23P_x^{10}(t_{3,2y-2})^2 + 13(t_{3,x})^2(t_{3,2y-2})^2$
- 5) $9P_y^6(t_{3,2x-2})^2 = 23(36P_{x-1}^8)(t_{3,y+1})^2 + 13(t_{3,2x-2})^2(t_{3,y+1})^2$
- 6) $(6P_{y-1}^4)^2(t_{3,x+1})^2 = 23(3P_x^3)^2(t_{3,2y-2})^2 + 13(t_{3,x+1})^2(t_{3,2y-2})^2$

IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2 = 23x^2 + 13$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.



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