



# **iJRASET**

International Journal For Research in  
Applied Science and Engineering Technology



---

# **INTERNATIONAL JOURNAL FOR RESEARCH**

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

---

**Volume: 7      Issue: VI      Month of publication: June 2019**

**DOI: <http://doi.org/10.22214/ijraset.2019.6284>**

**[www.ijraset.com](http://www.ijraset.com)**

**Call:  08813907089**

**E-mail ID: [ijraset@gmail.com](mailto:ijraset@gmail.com)**

# Numerical Results of Sub-Cauchy Problem for Linear Elasticity

Ali Nezam<sup>1</sup>, S. Moazzam Hussain<sup>2</sup>, Quaiser Suhail<sup>3</sup>

<sup>1,2,3</sup>Assistant Professor, Department of Mechanical Engineering, Maulana Azad College of Engineering And Technology, Neoraganj, P.O. Neora, Dist. Patna, Bihar, India 801113

**Abstract:** This work focuses on the sub-Cauchy problem for linear elasticity in two dimensional case. Solving such a problem may be formulated as follows: given the displacement and one component of the traction in a given part of the boundary of the elastic body, reconstruct the displacement field in all the domain. Author propose herein, an iterative method borrowed from the domain decomposition community to solve the sub-Cauchy problem. Numerical results highlight the efficiency of the proposed method.

**Keywords:** Linear elasticity; Shear stress; Cauchy problem; Steklov Poincare operator; Domain decomposition; Inverse problems

## I. INTRODUCTION

Many inverse problems in linear elasticity are defined by overdetermined boundary conditions. One can think to the reconstruction of buried flaws such as cracks, voids or inhomogeneities, identification of constitutive law, data completion (that is the recovery of boundary conditions on an inaccessible part of the body boundary) [1]. All the above inverse problems have in common to be defined by overspecified boundary conditions namely the normal stress and the displacement on a part of the boundary which correspond to Cauchy data. Many papers treated this problem, from the numerical view point, this last decade [2-4].

Author would like to mention the work by Bourgeois [5] who applied the Lions-quasi-reversibility method to the data completion. This method leads to a direct inversion process.

Many authors resort to iterative methods based on minimising a least-square type error functional, [6-8]. Marin [9] would like to mention the minimization of an energy-like gap functional in ref. [10] and domain decomposition like method in ref. [11] which are close to what we develop in this work.

Hereafter, Author are concerned by a partially overdetermined boundary conditions. In fact, on a part of the boundary of the domain partially overdetermined boundary data are prescribed, namely one component of the traction and the displacement field. Following ref. [12] author build an energy-gap error functional to recover the lacking boundary data. Author emphasise on the shear stress reconstruction, on the part of boundary where the partial-data is prescribed.

## II. FORMULATION OF SUB-CAUCHY PROBLEM AS STEKLOV POINCARE OPERATOR

The inverse problem under consideration concerns the recovery of lacking boundary data from the knowledge of partially overdetermined boundary elastic data. The problem is formulated mathematically as follows : Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , the boundary  $\Gamma = \partial\Omega$  is split into  $\Gamma_c$  and  $\Gamma_i$  having both non vanishing measure  $\Gamma_c \cap \Gamma_i = \emptyset$ . Given the displacement  $U$  and the normal component of surface traction  $\Phi.n$  on  $\Gamma_c$ :

$$\begin{cases} \operatorname{div} \sigma(u) = 0 & \text{in } \Omega \\ (\sigma(u) n).n = \Phi.n & \text{on } \Gamma_c \\ u = U & \text{on } \Gamma_i \end{cases} \quad (1)$$

where  $\sigma = \lambda \operatorname{Tr} \varepsilon(u) + 2\mu \varepsilon(u)$ ,  $\varepsilon = 1/2(\nabla u + \nabla u^T)$  and  $\lambda, \mu$  are the Lamé coefficients related to Young's modulus  $E$  and the Poisson ratio  $\nu$  via:

$$\mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

Our aim is then to reconstruct  $(\sigma(u).n).\tau$  on  $\Gamma_c$  and both the displacement and traction. To our knowledge, there are no theoretical studies (existence and uniqueness) of this problem despite its great importance in applications. In this paper author treat this problem numerically by solving a data completion problem.

The decomposition of the **Cauchy problem** (1) is formulated through an unknown function  $\eta$  as follows:

$$(P_D) \begin{cases} \text{div} \sigma(u_D) = 0 & \text{in } \Omega \\ u_D = U & \text{on } \Gamma_e \\ u_D = \eta & \text{on } \Gamma_l \end{cases} \quad (P_N) \begin{cases} \text{div} \sigma(u_N) = 0 & \text{in } \Omega \\ (\sigma(u_N).n).n = \Phi.n & \text{on } \Gamma_e \\ u_N.\tau = U.\tau & \text{on } \Gamma_e \\ u_N = \eta & \text{on } \Gamma_l \end{cases} \quad (2)$$

Where  $\eta$  is the virtual control and  $u$  is chosen so that  $u_D$  and  $u_N$  adjust in the best possible on  $\Omega$ . The solution  $u_D$  and  $u_N$  are a function of  $\eta$  ( $u_D = u_D(\eta)$  and  $u_N = u_N(\eta)$ ).

To express the problem in the framework of virtual control, we introduce the cost functional:

$$J(\eta) = \int_{\Omega} \sigma(u_D - u_N) : \varepsilon(u_D - u_N) \, d\Omega \quad (3)$$

and consider the minimization problem:

$$\inf_{\eta \in H^{\frac{1}{2}}(\Gamma_l)} J(\eta) \quad (4)$$

The solutions  $u_D$  and  $u_N$  can be written as:

$$u_D = u_D^0 + u_D^* \quad u_N = u_N^0 + u_N^*$$

Where  $u_i^0$  depends on the data  $U$  and  $\Phi.n$  where as  $u_i^*$  depends on  $\eta$  as follows:

$$(P_D^*) \begin{cases} \text{div} \sigma(u_D^*) = 0 & \text{in } \Omega \\ u_D^* = 0 & \text{on } \Gamma_e \\ u_D^* = \eta & \text{on } \Gamma_l \end{cases} \quad (P_D^0) \begin{cases} \text{div} \sigma(u_D^0) = 0 & \text{in } \Omega \\ u_D^0 = U & \text{on } \Gamma_e \\ u_D^0 = 0 & \text{on } \Gamma_l \end{cases} \quad (5)$$

Similarly, author decompose  $u_N$ .

$$(P_N^*) \begin{cases} \text{div} \sigma(u_N^*) = 0 & \text{in } \Omega \\ (\sigma(u_N^*).n).n = 0 & \text{on } \Gamma_e \\ u_N^*.\tau = 0 & \text{on } \Gamma_e \\ u_N^* = \eta & \text{on } \Gamma_l \end{cases} \quad (P_N^0) \begin{cases} \text{div} \sigma(u_N^0) = 0 & \text{in } \Omega \\ (\sigma(u_N^0).n).n = \Phi.n & \text{on } \Gamma_e \\ u_N^0.\tau = U.\tau & \text{on } \Gamma_e \\ u_N^0 = 0 & \text{on } \Gamma_l \end{cases} \quad (6)$$

The solution of the problem (4) is recovered if:

$$\sigma(u_D).n = \sigma(u_N).n \quad \text{on } \Gamma_l \quad (7)$$

With this partition, condition 7 leads to the boundary equation

$$\sigma(u_D^*) \cdot n - \sigma(u_N^*) \cdot n = \sigma(u_N^0) \cdot n - \sigma(u_D^0) \cdot n \quad \text{on } \Gamma_i \tag{8}$$

Author introduce the Steklov Poincaré operator

$$S\eta = \sigma(u_D^*) \cdot n - \sigma(u_N^*) \cdot n \quad \text{on } \Gamma_i$$

Author define  $S_D\eta = \sigma(u_D^*) \cdot n$  and  $S_N\eta = \sigma(u_N^*) \cdot n$ .

Author can write the equation (8) according to the Steklov Poincaré operator:

$$S\eta = \xi \quad \text{on } \Gamma_i$$

where  $\xi = -(\sigma(u_D^0) \cdot n - \sigma(u_N^0) \cdot n)$

This operator, borrowed from the domain decomposition community, is widely used in ref. [13].

There are several ways to solve this linear system of equations. Here author use an iterative preconditioned gradient algorithm, which appears to be very efficient. Each iteration of the algorithm is written

$$\eta = \eta + \rho(S_D)^{-1}(S\eta - \xi),$$

where  $\rho$  is a relaxation coefficient and  $S_D$  is the preconditioning operator.

Thus each iteration requires to compute  $S\eta$  by solving the two problems ?? and to solve the system  $S_D\chi = S\eta$ . This is achieved by solving the following problem:

$$\begin{cases} \text{div } \sigma(w) = 0 & \text{in } \Omega, \\ \sigma(w) \cdot n = S\eta & \text{on } \Gamma_i, \\ w = 0 & \text{on } \Gamma_e. \end{cases} \tag{9}$$

where  $\chi = w$  on  $\Gamma_i$ .

Now, author propose an algorithm to approximately solve the sub- Cauchy problem:

**Algorithm**

1. Choose arbitrary  $\eta$
2. Solve problems  $(P_D)$  and  $(P_N)$ .
3. solve problem (9).
4. Let  $\eta = \eta + \rho w$
5. Go back to the first step until the stopping criteria  $\|u_D - u_N\| \leq \epsilon$  is reached. ( $\epsilon$  is a given tolerance level)

**III. NUMERICAL RESULTS AND DISCUSSION**

The purpose of this section is to present the numerical implementation of the boundary data recovery process described above.

The numerical implementation is run under FreeFem software [14] based on Finite Element Method. All through this section, author consider an isotropic linear elastic material (Steel XC10 to 20° temperature) characterised by the poisson coefficient  $\nu = 0.29$  and Young's modulus  $E = 216$  GPa.

Author are concerned by a two dimensional framework corresponding to a **square hole domain**.

The partially overspecified boundary data is a synthetic one, obtained through the resolution of the following forward problem:

$$\begin{cases} \operatorname{div} \sigma(u_0) = 0 & \text{in } \Omega, \\ \sigma(u_0) \cdot n = \sigma(T) \cdot n & \text{on } \Gamma_c, \\ u_0 = T & \text{on } \Gamma_i. \end{cases}$$

where  $T = (\operatorname{Re}(\frac{1}{z-a}), \operatorname{Im}(\frac{1}{z-a}))$ ,  $z = x + iy$ ,  $a = 1.8$ ,  $\partial\Omega = \Gamma_c \cup \Gamma_i$  and  $\Gamma_i$  being the inner circle.

Notice that we are dealing with a “rough” case, insofar as, the inferred data, are induced by a “near singular” data. The trials used in the litterature come usually from analytical reference solutions.

#### A. Preliminary Numerical Test

Our trial concerns the resolution of the sub-Cauchy problem in the following context: We consider a square hole domain: rectangle size: (10 \* 20) with inner circle of radius  $R=2$ . The internal circle plays the role of the boundary  $\Gamma_i$  and the Cauchy data are donated in the **external boundary**  $\Gamma_c$ .

Author choose  $\epsilon = 10^{-2}$  in the stopping criteria computation are carried out with “un-noisy” data. **Figures 1-3** show the reconstructed displacement and traction on the inner boundary, whereas **Figure 4** illustrate the reconstruction of the shear stress in  $\Gamma_c$ . Note that the reconstruction is quite nice in  $\Gamma_i$  and in good agreement with exact for the shear stress.

#### B. Sensitivity to the Thickness

The following numerical trials are devoted to the influence of the radius of the hole on the reconstructed data. The results are summerized. As expected the computed sub-Cauchy problem solution is better when the distance between  $\Gamma_c$  and  $\Gamma_i$  is lower. To confirm the results, **Figure 5** where author present the result for the first component of displacement on  $\Gamma_i$

The same remark is true when we zoom on the shear stress.

#### C. Extended Domain

The following numerical experiments are inspired by Hecht [14]. In ref. [14], the authors resort to an extended domain method to illustrate its regularisation effect on their numerical data completion procedure. Their study was conducted in the framework of Laplace equation. To our opinion, the proposed method may be used in many practical situations, one can think to the data completion on rough boundary in this situation it is worthfull to extend the domain to a smooth one and to deduce the boundary conditions on the rough boundary. This trick will avoid the meshing difficulties for instance.

Another possible application may concern a void detection: If one has an apriori knowledge on the void location, the computation may be done on an extended domain, the void being detected by level lines of the displacement field [15] (for the Laplace equation). Our concern here is to illustrate the deblurring effect of this domain extension procedure.

Of course, it currently happens in practice that the data (on  $\Gamma_c$ ) suffer from erroneous measurements, the following numerical experiments illustrates the deblurring effect of the extended domain method.

We consider a random noise of 4% added to the exact data as follows:

$$U = \tilde{U} + \alpha r \quad \Phi \cdot n = \tilde{\Phi} \cdot n + \beta r$$

where  $(\alpha, \beta)$  denotes the noise level relative to  $(\|U\|_{L^2(\Gamma_c)}, \|\phi_m\|_{L^2(\Gamma_c)})$ , and  $r$  is a random function generated by Freefem.

The boundary  $\Gamma_i$  is very close to the complete boundary and is then exposed to the noise contamination coming from  $\Gamma_c$ . The possibility of extending domain by a fictitious incomplete boundary can correct this contamination.

The exact domain is defined square by rectangle size  $(10 * 20)$  with hole of radius  $R=6$  while the extended field is defined by the same rectangle, but with a hole of radius  $R=4$ .

Figures 6 and 7 show the reconstructed displacement for the exact and extended domain. Note that the solution computed in the real domain suffers from hard oscillations. Those obtained in the extended domain seem satisfactory.

#### IV. CONCLUSION

In this work the reconstruction of lacking boundary data on a part of the boundary of a body from partially-overspecified boundary conditions on another part has been investigated numerically. A domain decomposition like method has been given to describe the reconstruction process. The numerical investigation has been conducted on a “rough” configuration (i.e. the data to be recovered is not extendable on a divergence free stress field outside the domain namely within the hole), it uses FEM. The numerical section highlights the accuracy of the inverse procedure, as well as the **robustness** of the inversion process to noisy data as well as its ability to deblur noise.

#### REFERENCES

- [1] Bonnet M (2005) Inverse problems in elasticity. Inverse Problems 21 R1
- [2] Baranger TN, Andrieux S (2008) An optimization approach for the Cauchy problem in linear elasticity. Structural and Multidisciplinary Optimization 35: 141-152
- [3] Durand B, Delvare F, Bailly P (2011) Numerical solution of cauchy problems in linear elasticity. Int J Solids and Structures 48: 3041-3053
- [4] Bilotta A, Turco E (2009) A numerical study on the solution of the cauchy problem in elasticity. Int J Solids and Structures 46: 4451-4477
- [5] Bourgeois L, Dardé J (2010) Quasi-reversibility approach to solve the inverse obstacle problem. Inverse Probl Imaging 4: 351-377
- [6] Karageorghis D, Lesnic D, Marin L (2014) The method of fundamental solutions for an inverse boundary value problem in static thermo-elasticity. Computers and Structures 135: 32-39
- [7] Belgacem FB, Fekih HE (2005) On cauchy problem: I. A variational Steklov-Poincaré theory. Inverse Problems 21: 1915-1936
- [8] Hadamard J (1953) Lectures on cauchy's problem in linear partial differential equations. Dover New York USA
- [9] Marin L, Delvare F, Cimetière A (2015) Fading regularization MFS algorithm for inverse boundary value problems in two-dimensional linear elasticity. International Journal of Solids and Structures 78: 9-20
- [10] Andrieux S, Baranger TN (2008) An energy error-based method for the resolution of the Cauchy problem in 3D linear elasticity Comput. Methods Appl Mech Engrg 197: 902-920.
- [11] Kadri ML (2015) Contact pressures and cracks identification by using the Dirichlet-to-Neumann Solver in Elasticity. J Appl Mech Eng 4: 152
- [12] Andrieux S, Baranger T, Ben Abda (2006) Solving cauchy problems by minimizing an energy-like functional. Inverse Problems 22: 115-133
- [13] Quarteroni, Valli A (1999) Domain decomposition methods for partial differential equations. Oxford New York Clarendon press.
- [14] Hecht F, Pironneau O, Le Hyaric A, Ohtsuka K (2011) "FreeFem++" Univ Pierre et Marie Curie Paris
- [15] Ben Belgacem F, Du DT, Jlassi F. extended-domain-Lavrentiev's regularization for the cauchy problem. Inverse problems journal.

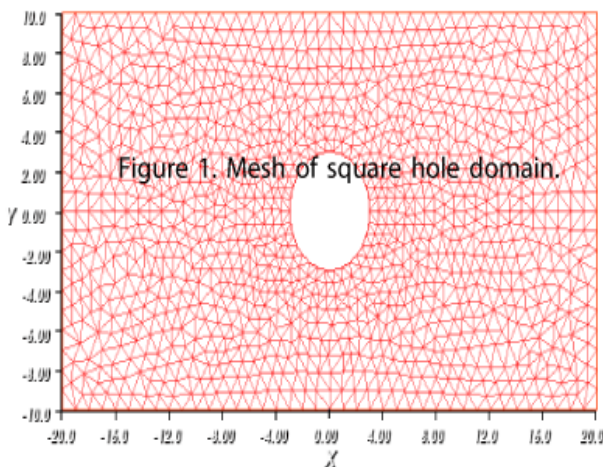
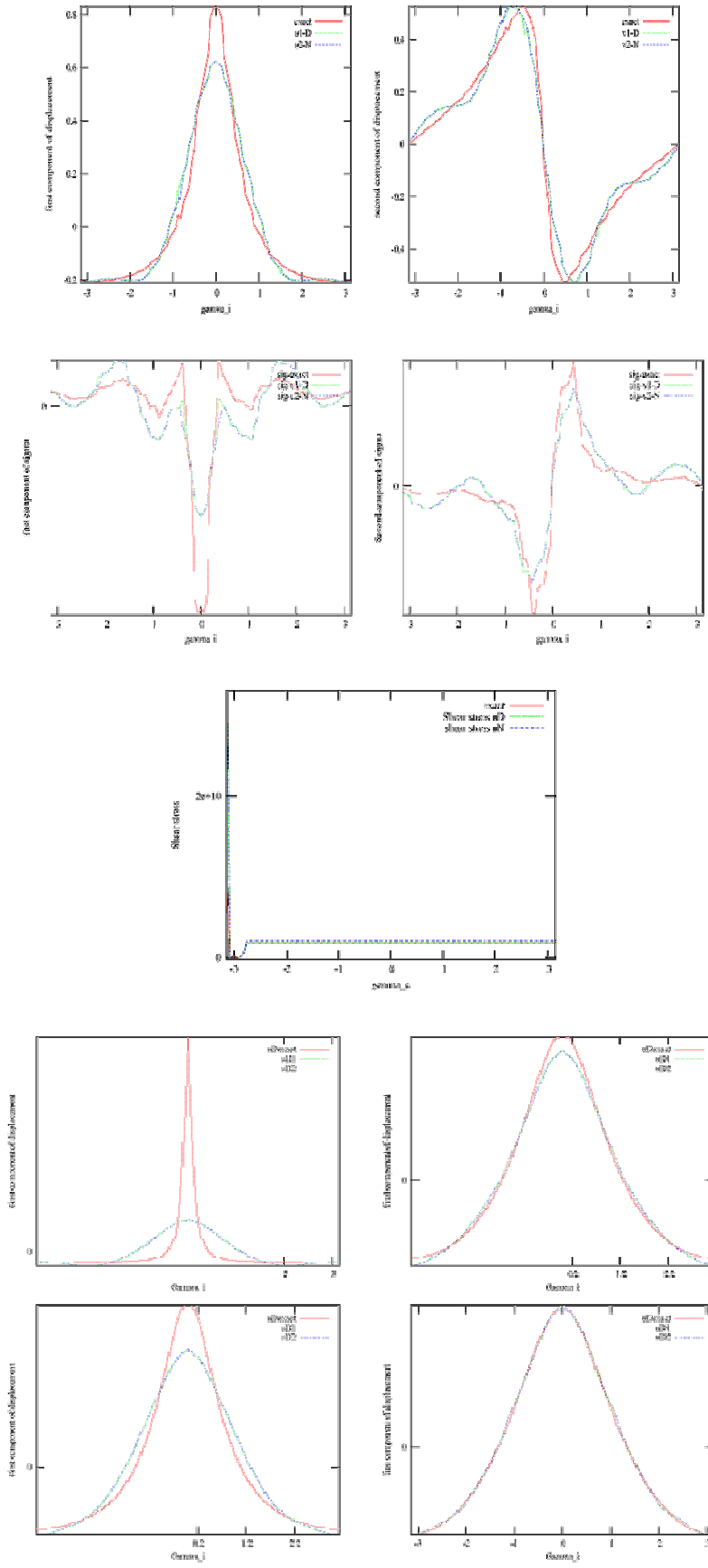
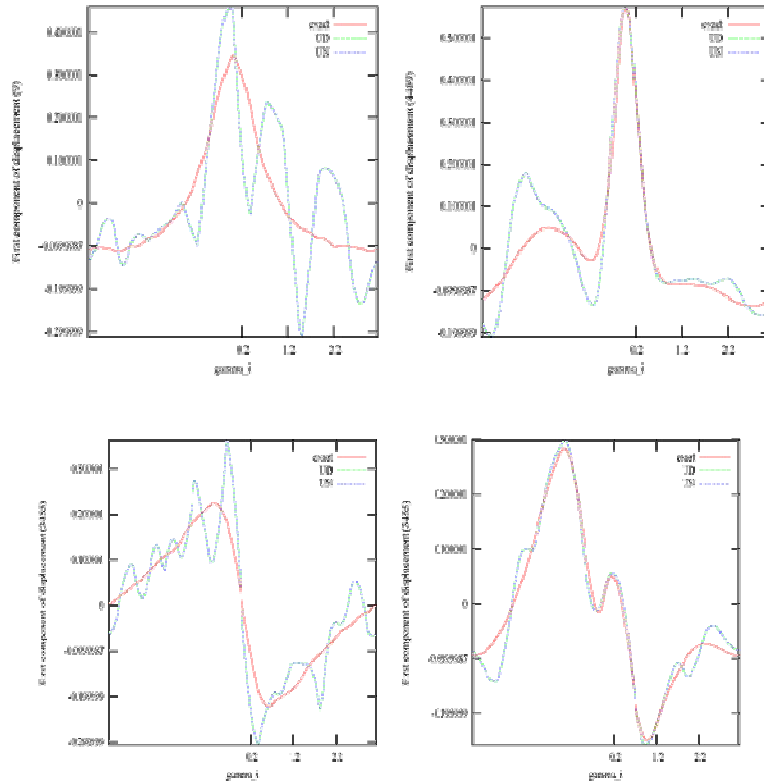


Figure 1. Mesh of square hole domain.









10.22214/IJRASET



45.98



IMPACT FACTOR:  
7.129



IMPACT FACTOR:  
7.429



# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24\*7 Support on Whatsapp)