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Existence of Fixed Point Theorems in Linear 2-Normed Spaces

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Abstract–In this paper, we discuss the concept of some existence of fixed point theorems for a pair of mappings in a setting of 2-normed spaces.

Keywords–Normed space, 2-normed space, Fixed point, Common fixed point.

I. INTRODUCTION

In 1963, S.Gahler [5] was initiated by 2-normed space & 2-Banach space. In 1975, K.Iseki was discussed the concept of the fundamental results on fixed point theorems of 2-metric space. Next Albert White and Y.J.Cho [1] investigate the important properties of linear mappings on linear 2-normed space in the year of 1984. Here after, many authors establish the fixed point theorem in 2-normed spaces and 2-Banach spaces. See [4,6,16–20]. Recently, generalise the concept of 2-normed space as well as 2-Banach space into 2-cone Banach space [3,12,22].

We now state some definitions before presenting our main results.

II. PRELIMINARIES

1) *Definition 1.* [4] Let X be a real linear space with dimension of X is greater than 1 and $\| \dots \| : X \times X \rightarrow [0, \infty)$ be a function. Then

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent,

(ii) $\| x, y \| = \| y, x \|$,

(iii) $\| \alpha x, y \| = |\alpha| \| x, y \|$,

(iv) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$, where for all $x, y, z \in X$ and $\alpha \in R$.

If $\| \dots \|$ is called a 2-norm and the pair $(X, \| \dots \|)$ is called a linear 2-normed space. So a 2-norm $\| x, y \|$ always satisfies [24] $\| x, y + \alpha x \| = \| x, y \|$ for all $x, y \in X$ and all scalars α .

2) *Definition 2.* [4] A sequence $\{x_n\}$ in X is convergent to an element $x \in X$, if for each $a \in X$, $\lim_{n \rightarrow \infty} \| x_n - x, a \| = 0$. If $\{x_n\}$ converges to x we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

3) *Definition 3.* [4] A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $a \in X$, $\lim_{n, m \rightarrow \infty} \| x_n - x_m, a \| = 0$ as $n, m \rightarrow \infty$.

4) *Definition 4.* [4] A complete 2-normed space is one in which every Cauchy sequence in X converges to an element of X . A complete 2-normed space X is called 2-Banach space.

5) *Definition 5.* [16] Let $(X, \| \dots \|)$ be a linear 2-normed space. Then the mapping $T : X \rightarrow X$ is said to be a contraction if there exists $k \in [0, 1)$ such that $\| Tx - Ty, z \| \leq k \| x - y, z \|$ for all $x, y, z \in X$.

6) *Definition 6.* [11] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

a) ψ is non-decreasing and continuous,

b) $\psi(t) = 0$ if and only if $t = 0$.

7) *Definition 7.* [14] An ultra-altering distance function is a continuous, non-decreasing mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) > 0, t \in [0, \infty)$ and $\varphi(0) \geq 0$.

We denote this set with Φ_u .

Delbosco [2] and Skof [23] have proved a fixed point theorem for self-maps of complete normed spaces by introducing a class Φ of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- a) $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous in R^+ and strictly increasing in R^+ .
- b) $\phi(t) = 0$ if and only if $t = 0$.
- c) $\phi(t) \geq Mt^\mu$ for every $t > 0, \mu > 0$ are constants.

III. MAIN RESULT

1) *Theorem 1.* Let T be a self-map of a complete 2-normed space $(X, \|\cdot, \cdot\|)$ and ϕ satisfying (i) and (ii).

Furthermore, let f, g, h be three decreasing functions from R^+ into $[0,1)$ such that $f(t) + 2g(t) + h(t) < 1$ for every $t > 0$. Suppose T satisfies the following condition

$$\begin{aligned} \phi(\|Tx - Ty, u\|) \leq & f(\|x - y, u\|)\phi(\|x - y, u\|) + g(\|x - y, u\|)[\phi(\|x - T_x, u\|) + \phi(\|y - T_y, u\|)] \\ & + h(\|x - y, u\|)\min\{\phi(\|x - T_x, u\|), \phi(\|y - T_x, u\|)\} \end{aligned} \quad (1)$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then T has an unique fixed point.

Proof. Let us take x_0 be arbitrary point in X .

define $x_{n+1} = Tx_n; n = 0, 1, 2, \dots$ and $\alpha_n = \|x_n - x_{n+1}, u\|$ for $n = 0, 1, 2, \dots$; and $\beta_n = \phi(\alpha_n)$.

Then we have

$$\begin{aligned} \beta_{n+1} &= \phi(\alpha_{n+1}) \\ &= \phi(\|x_{n+1} - x_{n+2}, u\|) \\ &= \phi(\|Tx_n - Tx_{n+1}, u\|) \\ &\leq f(\|x_n - x_{n+1}, u\|)\phi(\|x_n - x_{n+1}, u\|) + g(\|x_n - x_{n+1}, u\|)[\phi(\|x_n - Tx_n, u\|) + \phi(\|x_{n+1} - Tx_{n+1}, u\|)] \\ &\quad + h(\|x_n - x_{n+1}, u\|)\min\{\phi(\|x_n - Tx_{n+1}, u\|), \phi(\|x_{n+1} - Tx_n, u\|)\} \\ &= f(\|x_n - x_{n+1}, u\|)\phi(\|x_n - x_{n+1}, u\|) + g(\|x_n - x_{n+1}, u\|)[\phi(\|x_n - x_{n+1}, u\|) + \phi(\|x_{n+1} - x_{n+2}, u\|)] \\ &\quad + h(\|x_n - x_{n+1}, u\|)\min\{\phi(\|x_n - x_{n+2}, u\|), \phi(\|x_{n+1} - x_{n+1}, u\|)\} \\ &= f(\alpha_n)\phi(\alpha_n) + g(\alpha_n)[\phi(\alpha_n) + \phi(\alpha_{n+1})] \end{aligned} \quad (2)$$

$$\text{implies } \beta_{n+1} \leq \frac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)} \beta_n.$$

Since $f(t) + 2g(t) + h(t) < 1$, $f(\alpha_n) + 2g(\alpha_n) < 1$ which implies $\frac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)} < 1$.

If we set $r = \frac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)}$ then from (2) we get $\beta_{n+1} \leq r\beta_n$ where $r < 1$.

So $\beta_n \leq r^n \beta_0$, such that $\beta_n \rightarrow 0$ as $n \rightarrow \infty$.

Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$, $n = 1, 2, \dots$.

Thus $\alpha_n \rightarrow \alpha$. Then $\beta_n = \phi(\alpha_n) \rightarrow \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \rightarrow 0$.

Now show that $\{x_n\}$ is a Cauchy sequence.

We prove it by contradiction. Then for every positive integer ε and for every positive integer k there exist two positive integers $m(k)$ and $n(k)$ such that

$$k < n(k) < m(k) \text{ and } \|x_{m(k)} - x_{n(k)}, u\| > \varepsilon \tag{3}$$

For each integer k let $m(k)$ be the least integer for which $m(k) > n(k) > k$,

$$\|x_{n(k)} - x_{m(k)-1}, u\| \leq \varepsilon \text{ and } \|x_{n(k)} - x_{m(k)}, u\| > \varepsilon$$

Then we have

$$\begin{aligned} \varepsilon &< \|x_{n(k)} - x_{m(k)}, u\| \\ &\leq \|x_{n(k)} - x_{m(k)}, x_{m(k)-1}\| + \|x_{n(k)} - x_{m(k)-1}, u\| + \|x_{m(k)-1} - x_{m(k)}, u\| \end{aligned} \tag{4}$$

Now by (1), we have

$$\begin{aligned} \phi(\|x_{n(k)} - x_{m(k)}, x_{m(k)-1}\|) &= \phi(\|Tx_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1}\|) \\ &\leq f(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \phi(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \\ &+ g(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) [\phi(\|x_{n(k)-1} - Tx_{n(k)-1}, x_{m(k)-1}\|) + \phi(\|x_{m(k)-1} - Tx_{m(k)-1}, x_{m(k)-1}\|)] \\ &+ h(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \min\{\phi(\|x_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1}\|), \phi(\|x_{m(k)-1} - Tx_{n(k)-1}, x_{m(k)-1}\|)\} \\ &= f(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \phi(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \\ &+ g(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) [\phi(\|x_{n(k)-1} - Tx_{n(k)-1}, x_{m(k)-1}\|) + \phi(\|x_{m(k)-1} - Tx_{m(k)-1}, x_{m(k)-1}\|)] \\ &+ h(\|x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1}\|) \min\{\phi(\|x_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1}\|), \phi(\|x_{m(k)-1} - Tx_{n(k)-1}, x_{m(k)-1}\|)\} \\ &= 0 \end{aligned}$$

which implies by (ii) $\|x_{n(k)} - x_{m(k)}, x_{m(k)-1}\| = 0$. (5)

So by (4) and (5) we get, $\varepsilon < \|x_{n(k)} - x_{m(k)}, u\| \leq 0 + \varepsilon + \alpha_{m(k)-1}$. Since $\{\alpha_n\}$ converges to 0,

$$\|x_{n(k)} - x_{m(k)}, u\| \rightarrow \varepsilon \text{ as } k \rightarrow \infty.$$

Again

$$\begin{aligned} \|x_{n(k)+1} - x_{m(k)}, u\| &\leq \|x_{n(k)+1} - x_{m(k)}, x_{n(k)}\| + \|x_{n(k)+1} - x_{n(k)}, u\| + \|x_{n(k)} - x_{m(k)}, u\| \\ &\leq \alpha_{n(k)} + \|x_{n(k)} - x_{m(k)}, u\|, \end{aligned}$$

since, $\|x_{n(k)+1} - x_{m(k)}, x_{n(k)}\|$ can be made 0 as we have done in equation (5).

So $\|x_{n(k)+1} - x_{m(k)}, u\| \leq \alpha_{n(k)} + \|x_{n(k)} - x_{m(k)}, u\| \rightarrow \varepsilon$ as $k \rightarrow \infty$.

In the similar way,

$$\begin{aligned} \|x_{n(k)+2} - x_{m(k)}, u\| &\leq \|x_{n(k)+2} - x_{m(k)}, x_{n(k)+1}\| + \|x_{n(k)+2} - x_{n(k)+1}, u\| + \|x_{n(k)+1} - x_{m(k)}, u\| \\ &\leq \alpha_{n(k)+1} + \|x_{n(k)+1} - x_{m(k)}, u\|, \end{aligned}$$

since $\|x_{n(k)+2} - x_{m(k)}, x_{n(k)+1}\|$ can be made 0 as we have done in equation (5).

So $\|x_{n(k)+2} - x_{m(k)}, u\| \leq \alpha_{n(k)+1} + \|x_{n(k)+1} - x_{m(k)}, u\| \rightarrow \varepsilon$ as $k \rightarrow \infty$ and in similar fashion we can show

$$\|x_{n(k)+2} - x_{m(k)+1}, u\| \rightarrow \varepsilon \text{ as } k \rightarrow \infty.$$

Using (1), we deduce that

$$\begin{aligned} \phi(\|x_{n(k)+2} - x_{m(k)+1}, u\|) &= \phi(\|Tx_{n(k)+1} - Tx_{m(k)}, u\|) \\ &\leq f(\|x_{n(k)+1} - x_{m(k)}, u\|)\phi(\|x_{n(k)+1} - x_{m(k)}, u\|) \\ &\quad + g(\|x_{n(k)+1} - x_{m(k)}, u\|)[\phi(\|x_{n(k)+1} - Tx_{n(k)+1}, u\|) + \phi(\|x_{m(k)} - Tx_{m(k)}, u\|)] \\ &\quad + h(\|x_{n(k)+1} - x_{m(k)}, u\|)\min\{\phi(\|x_{n(k)+1} - Tx_{m(k)}, u\|), \phi(\|x_{m(k)} - Tx_{n(k)+1}, u\|)\} \\ &\leq f(\|x_{n(k)+1} - x_{m(k)}, u\|)\phi(\|x_{n(k)+1} - x_{m(k)}, u\|) \\ &\quad + g(\|x_{n(k)+1} - x_{m(k)}, u\|)[\phi(\|x_{n(k)+1} - x_{n(k)+2}, u\|) + \phi(\|x_{m(k)} - x_{m(k)+1}, u\|)] \\ &\quad + h(\|x_{n(k)+1} - x_{m(k)}, u\|)\min\{\phi(\|x_{n(k)+1} - x_{m(k)+1}, u\|), \phi(\|x_{m(k)} - x_{n(k)+2}, u\|)\} \end{aligned}$$

Letting $k \rightarrow \infty$, we get $\phi(\varepsilon) \leq a(\varepsilon)\phi(\varepsilon) + c(\varepsilon)\phi(\varepsilon) = \{a(\varepsilon) + c(\varepsilon)\}\phi(\varepsilon) < \phi(\varepsilon)$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete 2-normed space, $\lim_{n \rightarrow \infty} x_n = z \in X$.

Claim: Show that $Tz = z$.

Again using (1) we have,

$$\begin{aligned} \phi(\|x_{n(k)+1} - Tz, u\|) &= \phi(\|Tx_{n(k)} - Tz, u\|) \\ &\leq f(\|x_{n(k)} - z, u\|)\phi(\|x_{n(k)} - z, u\|) \\ &\quad + g(\|x_{n(k)} - z, u\|)[\phi(\|x_{n(k)} - Tx_{n(k)}, u\|) + \phi(\|z - Tz, u\|)] \\ &\quad + h(\|x_{n(k)} - z, u\|)\min\{\phi(\|x_{n(k)} - Tz, u\|), \phi(\|z - Tx_{n(k)}, u\|)\} \end{aligned}$$

implies

$$\begin{aligned} \phi(\|x_{n(k)+1} - Tz, u\|) &\leq f(\|x_{n(k)} - z, u\|)\phi(\|x_{n(k)+1} - z, u\|) \\ &\quad + g(\|x_{n(k)} - z, u\|)[\phi(\|x_{n(k)} - x_{n(k)+1}, u\|) + \phi(\|z - Tz, u\|)] \\ &\quad + h(\|x_{n(k)} - z, u\|)\min\{\phi(\|x_{n(k)} - Tz, u\|), \phi(\|z - x_{n(k)+1}, u\|)\} \end{aligned}$$

passing limit as $n \rightarrow \infty$ on both sides of the inequality we get, $\phi(\|z - Tz, u\|) = 0$ which gives by (ii), $\|z - Tz, u\| = 0$ i.e., $Tz = z$.

Next let w be another fixed point of T . Then

$$\begin{aligned} \phi(\|z - w, u\|) &= \phi(\|Tz - Tw, u\|) \\ &\leq f(\|z - w, u\|)\phi(\|z - w, u\|) + g(\|z - w, u\|)[\phi(\|z - Tz, u\|) + \phi(\|w - Tw, u\|)] \\ &\quad + h(\|z - w, u\|)\min\{\phi(\|z - Tw, u\|), \phi(\|w - Tz, u\|)\} \\ &= [f(\|z - w, u\|) + h(\|z - w, u\|)]\phi(\|z - w, u\|) \\ &< \phi(\|z - w, u\|), \text{ Since } f(t) + h(t) < 1 \end{aligned}$$

which is a contradiction leads to the fact that $z = w$ and thus completes the proof.

Next we verify the Theorem (1) by a proper example.

a) *Example 1.* Let $X = R^+ \times R^+$ and d be a 2-normed which expresses $\|x - y, u\|$ as the area of the Euclidean triangle with vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $u = (u_1, u_2)$. Then $(X, \|\cdot, \cdot\|)$ is a complete 2-normed space[1].

Now take $x = (1,0)$, $y = (2,0)$ and $u = (1,1)$ also let $T : X \rightarrow X$ be a mapping such that

$$Tx = (2,0) \text{ where } x = (1,0) \in X \text{ and}$$

$$Ty = (3,0) \text{ where } y = (2,0) \in X$$

Now setting $f(t) = \frac{2}{5}$, $g(t) = \frac{1}{5}$, $h(t) = \frac{1}{6}$ and $\phi(t) = t^2; t \in R^+$.

We observe that all the conditions of Theorem (1) satisfied except the condition (1). Also it is very clear that T has no fixed point in X in this case.

Next we establish a common fixed point theorem in this line.

2) *Theorem 2.* Let S and T be self-mappings of a complete 2-normed space $(X, \|\cdot, \cdot\|)$ and ϕ satisfying (i) and (ii).

Furthermore, let f, g, h be three decreasing functions from R^+ into $[0,1)$ such that $f(t) + 2g(t) + h(t) < 1$ for every $t > 0$. Suppose S and T satisfies the following condition

$$\begin{aligned} \phi(\|Sx - Ty, u\|) &\leq f(\|x - y, u\|)\phi(\|x - y, u\|) \\ &\quad + g(\|x - y, u\|)[\phi(\|x - Sx, u\|) + \phi(\|y - Ty, u\|)] \\ &\quad + h(\|x - y, u\|)\min\{\phi(\|x - Ty, u\|), \phi(\|y - Sx, u\|)\} \end{aligned} \tag{6}$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary.

define $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$; $n = 0, 1, 2, \dots$, also let $\alpha_n = \|x_n - x_{n+1}, u\|$ for $n = 0, 1, 2, \dots$; and $\beta_n = \phi(\alpha_n)$. We also assume that $\alpha_n > 0$ for every n . Now for an even integer n , we have

$$\begin{aligned} \beta_n &= \phi(\alpha_n) \\ &= \phi(\|x_n - x_{n+1}, u\|) \\ &= \phi(\|Sx_{n-1} - Tx_n, u\|) \\ &\leq f(\|x_{n-1} - x_n, u\|)\phi(\|x_{n-1} - x_n, u\|) \\ &\quad + g(\|x_{n-1} - x_n, u\|)[\phi(\|x_{n-1} - Sx_{n-1}, u\|) + \phi(\|x_n - Tx_n, u\|)] \\ &\quad + h(\|x_{n-1} - x_n, u\|)\min\{\phi(\|x_{n-1} - Tx_n, u\|), \phi(\|x_n - Sx_{n-1}, u\|)\} \\ &= f(\|x_{n-1} - x_n, u\|)\phi(\|x_{n-1} - x_n, u\|) \\ &\quad + g(\|x_{n-1} - x_n, u\|)[\phi(\|x_{n-1} - x_n, u\|) + \phi(\|x_n - x_{n+1}, u\|)] \\ &\quad + h(\|x_{n-1} - x_n, u\|)\min\{\phi(\|x_{n-1} - x_{n+1}, u\|), \phi(\|x_n - x_n, u\|)\} \\ &= f(\alpha_{n-1})\phi(\alpha_{n-1}) + g(\alpha_{n-1})[\phi(\alpha_{n-1}) + \phi(\alpha_n)] \end{aligned}$$

$$\text{implies } \beta_n \leq \frac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})} \beta_{n-1} \tag{7}$$

Since $f(t) + 2g(t) + h(t) < 1$, $f(\alpha_{n-1}) + 2g(\alpha_{n-1}) < 1$ which implies $\frac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})} < 1$

$$\text{If we set } r = \frac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})}$$

then from (7) we get $\beta_n \leq r\beta_{n-1}$ where $r < 1$.

So $\beta_n \leq r^n \beta_0$, such that $\beta_n \rightarrow 0$ as $n \rightarrow \infty$.

Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$, $n = 1, 2, \dots$.

Thus $\alpha_n \rightarrow \alpha$. Then $\beta_n = \phi(\alpha_n) \rightarrow \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \rightarrow 0$.

Now show that $\{x_n\}$ is a Cauchy sequence.

We prove it by contradiction. Then for every positive integer ε and for every positive integer k there exist two positive integers $2p(k)$ and $2q(k)$ such that

$$k < 2q(k) < 2p(k) \text{ and } \|x_{2p(k)} - x_{2q(k)}, u\| > \varepsilon. \tag{8}$$

For each integer k let $2p(k)$ be the least integer for which $2p(k) > 2q(k) > k$,

$$\|x_{2q(k)} - x_{2p(k)-2}, u\| \leq \varepsilon \text{ and } \|x_{2q(k)} - x_{2p(k)}, u\| > \varepsilon$$

Then we have

$$\varepsilon < \|x_{2q(k)} - x_{2p(k)}, u\| \leq \|x_{2q(k)} - x_{2p(k)}, x_{2p(k)-2}\| + \|x_{2q(k)} - x_{2p(k)-2}, u\| + \|x_{2p(k)-2} - x_{2p(k)}, u\|$$

Since we can easily show that $\|x_{2q(k)} - x_{2p(k)}, x_{2p(k)-2}\| = 0$ as we have shown in (5) of Theorem (1).

$$\begin{aligned} \varepsilon < \|x_{2q(k)} - x_{2p(k)}, u\| &\leq \|x_{2q(k)} - x_{2p(k)-2}, u\| + \|x_{2p(k)-2} - x_{2p(k)}, u\| \\ &\leq \|x_{2q(k)} - x_{2p(k)-2}, u\| + \|x_{2p(k)-2} - x_{2p(k)}, x_{2p(k)-1}\| + \|x_{2p(k)-2} - x_{2p(k)-1}, u\| + \|x_{2p(k)-1} - x_{2p(k)}, u\| \end{aligned}$$

again we can show like (5) of Theorem (1),

$$\|x_{2p(k)-2} - x_{2p(k)}, x_{2p(k)-1}\| = 0.$$

Thus

$$\varepsilon < \|x_{2q(k)} - x_{2p(k)}, u\| \leq \varepsilon + 0 + \alpha_{2p(k)-2} + \alpha_{2p(k)-1}. \tag{9}$$

Since $\{\alpha_n\}$ converges to 0, $\|x_{2q(k)} - x_{2p(k)}, u\| \rightarrow \varepsilon$.

$$\begin{aligned} \text{Now } \|x_{2q(k)} - x_{2p(k)+1}, u\| &\leq \|x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)}\| + \|x_{2q(k)} - x_{2p(k)}, u\| + \|x_{2p(k)} - x_{2p(k)+1}, u\| \\ &\leq \|x_{2q(k)} - x_{2p(k)}, u\| + \alpha_{2p(k)} \end{aligned}$$

since we can show that $\|x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)}\| = 0$ as we have done in (5) of Theorem (1).

$$\text{So, } \|x_{2q(k)} - x_{2p(k)+1}, u\| \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \tag{10}$$

Again

$$\begin{aligned} \|x_{2q(k)} - x_{2p(k)+2}, u\| &\leq \|x_{2q(k)} - x_{2p(k)+2}, x_{2p(k)+1}\| + \|x_{2q(k)} - x_{2p(k)+1}, u\| + \|x_{2p(k)+1} - x_{2p(k)+2}, u\| \\ &\leq \|x_{2q(k)} - x_{2p(k)+1}, u\| + \|x_{2p(k)+1} - x_{2p(k)+2}, u\|, \end{aligned}$$

(since $\|x_{2q(k)} - x_{2p(k)+2}, x_{2p(k)+1}\| = 0$ for similar reason as of (5) of Theorem (1).)

$$\begin{aligned} &\leq \|x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)}\| + \|x_{2q(k)} - x_{2p(k)}, u\| + \|x_{2p(k)} - x_{2p(k)+1}, u\| \\ &\quad + \|x_{2p(k)+1} - x_{2p(k)+2}, u\| \end{aligned}$$

$$\leq 0 + \|x_{2q(k)} - x_{2p(k)}, u\| + \alpha_{2p(k)} + \alpha_{2p(k)+1}$$

$$\text{which gives } \|x_{2q(k)} - x_{2p(k)+2}, u\| \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \tag{11} \text{ Similarly,}$$

$$\|x_{2q(k)+1} - x_{2p(k)+2}, u\| \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \tag{12}$$

Now from (6) we have

$$\begin{aligned} \phi(\|x_{2p(k)+2} - x_{2q(k)+1}, u\|) &= \phi(\|Sx_{2p(k)+1} - Tx_{2q(k)}, u\|) \\ &\leq f(\|x_{2p(k)+1} - x_{2q(k)}, u\|) \phi(\|x_{2p(k)+1} - x_{2q(k)}, u\|) \\ &\quad + g(\|x_{2p(k)+1} - x_{2q(k)}, u\|) [\phi(\|x_{2p(k)+1} - Sx_{2p(k)+1}, u\|) + \phi(\|x_{2q(k)} - Tx_{2q(k)}, u\|)] \\ &\quad + h(\|x_{2p(k)+1} - x_{2q(k)}, u\|) \min \{ \phi(\|x_{2p(k)+1} - Tx_{2q(k)}, u\|), \phi(\|x_{2q(k)} - Sx_{2p(k)+1}, u\|) \} \end{aligned}$$

passing limit as $k \rightarrow \infty$ we get by (10), (11) and (12),

$$\phi(\varepsilon) \leq f(\varepsilon)\phi(\varepsilon) + h(\varepsilon)\phi(\varepsilon) = \{f(\varepsilon) + h(\varepsilon)\}\phi(\varepsilon) < \phi(\varepsilon)$$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete 2-normed space, $\lim_n x_n = z \in X$.

Again using (6) we have

$$\begin{aligned} & \phi(\|x_{2p(k)+2} - Tz, u\|) = \phi(\|Sx_{2p(k)+1} - Tz, u\|) \\ & \leq f(\|x_{2p(k)+1} - z, u\|)\phi(\|x_{2p(k)+1} - z, u\|) \\ & \quad + g(\|x_{2p(k)+1} - z, u\|)[\phi(\|x_{2p(k)+1} - Sx_{2p(k)+1}, u\|) + \phi(\|z - Tz, u\|)] \\ & \quad + h(\|x_{2p(k)+1} - z, u\|)\min\{\phi(\|x_{2p(k)+1} - Tz, u\|), \phi(\|z - Sx_{2p(k)+1}, u\|)\} \end{aligned}$$

Taking limit as $k \rightarrow \infty$ we get $\phi(\|z - Tz, u\|) = 0$ implies $\|z - Tz, u\| = 0$ by property (ii).

Hence $Tz = z$.

Similarly it can be shown that $Sz = z$. So S and T have a common fixed point $z \in X$.

We now show that z is the unique common fixed point of S and T .

If not, then let w be another fixed point of S and T .

$$\begin{aligned} & \text{Then } \phi(\|z - w, u\|) = \phi(\|Sz - Tw, u\|) \\ & \leq f(\|z - w, u\|)\phi(\|z - w, u\|) \\ & \quad + g(\|z - w, u\|)[\phi(\|z - Sz, u\|) + \phi(\|w - Tw, u\|)] \\ & \quad + h(\|z - w, u\|)\min\{\phi(\|z - Tw, u\|), \phi(\|w - Sz, u\|)\} \\ & = [f(\|z - w, u\|) + h(\|z - w, u\|)]\phi(\|z - w, u\|) \\ & < \phi(\|z - w, u\|), \text{ since } f(t) + h(t) < 1 \end{aligned}$$

which is a contradiction. Hence $z = w$ and thus completes the proof.

Remark 1. In the same way we can verify the Theorem (2) by setting $S(1,0) = (2,0)$ and $T(2,0) = (3,0)$ taking all the values same on the complete 2-normed space $(X, \|\cdot, \cdot\|)$ as described in Example 1.

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