



# IJRASET

International Journal For Research in  
Applied Science and Engineering Technology



---

# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

---

**Volume: 7      Issue: VII      Month of publication: July 2019**

**DOI: <http://doi.org/10.22214/ijraset.2019.7049>**

**[www.ijraset.com](http://www.ijraset.com)**

**Call:  08813907089**

**E-mail ID: [ijraset@gmail.com](mailto:ijraset@gmail.com)**

# To Study Separation Axioms in Topological Spaces

Meenu Bharti<sup>1</sup>, Prof. R.B. Singh<sup>2</sup>

<sup>1</sup>Master of Science In Mathematics, Monad University Hapur

<sup>2</sup>Department of Mathematics

**Abstract:** The purpose of this paper is to introduce weak separation axioms via *sgp*-closed sets in topological spaces and study some of their properties.

**Keywords:** T0- space or Kolmogorov Spaces, T1-Space or Frechet's Separation axioms, T2-Space or Hausdorff Space, T3- Space Regular spaces, T4-spaces or normal spaces.

## I. INTRODUCTION

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms.

The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of kinds of topological spaces. However, the term "separation axiom" has stuck. The separation axioms are denoted with the letter "T" after the German Trennungsaxiom, which means "separation axiom."

### A. T0- Spaces or Knowledge Spaces

- 1) **Definition:** A topological space  $(X, \tau)$  is said to be T0- space if and only if given any pair of distinct points  $x, y$  of  $X$ , there exists a neighbourhood of one of them not containing the other.
- 2) **Example 1:** Show that every discrete space is a T0- space.
- 3) **Solution:** Let  $(X, D)$  be a discrete topological space and let  $x, y$  be distinct points of  $X$ . Since the space is discrete,  $\{x\}$  is an open nhd of  $x$  which does not contain  $y$ . It follows that  $(X, D)$  is a T0 - space.
- 4) **Example 2:** If  $X = \{a, b\}$  and  $\tau = \{\Phi, \{a\}, X\}$ . Is the space  $(X, \tau)$  is T0 - space?
- 5) **Solution:**  $(X, \tau)$  is T0- space as for given  $a, b \in X \Rightarrow \exists \{a\} \in \tau$  such that  $a \in \{a\}, b \notin \{a\}$ .
- 6) **Theorem:** Every subspace of a T0 - space is a T0 - space and hence the property is hereditary.
- 7) **Proof:** Let  $(X, \tau)$  be T0 - space and let  $(X, \tau^*)$  be a subspace of  $(X, \tau)$ . Let  $y_1, y_2$  be two distinct points of  $Y$ . Since  $Y \subset X, y_1, y_2$  are also distinct points of  $X$ . Since  $(X, \tau)$  is a T0 - space, there exists a  $\tau$ -open nhd  $G$  of  $y_1$  not containing  $y_2$ . Then  $G \cap Y$  is  $\tau^*$ -open nhd  $G$  of  $y_1$  not containing  $y_2$ . It follows that  $(X, \tau^*)$  is a T0 - space.

### B. T1 - Spaces or Frechet's Separation Axiom

**Definition:** A topological space  $(X, \tau)$  is said to be T1 space if and only if given any pair of distinct points  $x$  and  $y$  of  $X$  there exist two open sets one containing  $x$  but not  $y$ , and the other containing  $y$  but not  $x$ , that is, there exist open sets  $G$  and  $H$  such that  $x \in G$  but  $y \notin G$  and  $y \in H$  but  $x \notin H$ .

**Example 1:** Show that space  $(\mathbb{R}, U)$  is a T1 .

**Solution:** Let  $x, y$  be any two distinct real numbers and let  $y > x$ . Let  $y - x = k$ . Then  $G = ] x - k/4, x + k/4 [$  and  $H = ] y - k/4, y + k/4 [$  are  $U$ -open sets such that  $x \in G$  but  $y \notin G$  and  $y \in H$  but  $x \notin H$ . Hence  $(\mathbb{R}, U)$  is a T1 -space.

### C. T2 - Spaces or Hausdorff Space

**Definition :** A topology space  $(X, \tau)$  is said to be a T2- space (Hausdorff space or separated space) iff for every pair of distinct points  $x, y$  of  $X$ , there exists disjoint neighbourhoods of  $x$  and  $y$ , that is, there exist neighbourhoods  $N$  of  $x$  and  $M$  of  $y$  such that  $N \cap M = \Phi$ . If  $(X, \tau)$  is a Hausdorff space, then  $\tau$  is said to be a Hausdorff topology for  $X$ .

**Example 1:** Show that every discrete space is T2. Also show that no indiscrete space consisting of at least two points is Hausdorff.

**Solution :** Since every singleton set is open in a discrete space, it follows that every pair of distinct points of such a space has disjoint neighbourhoods, namely, the singleton sets containing these points. Hence every discrete space is Hausdorff. But no indiscrete space can be Hausdorff since the whole space is the only neighbourhood of each of its points so that no two distinct points can have disjoint neighbourhoods.

**D. T3 - Spaces or Regular Spaces**

1) **Definition :** A topology space  $(X, \tau)$  is said to be regular if and only if for every  $\tau$ -closed set  $F$  and every point  $p \notin F$ , there exist  $\tau$ -open sets  $G$  and  $H$  such that

$$F \subset G, p \in H \text{ and } G \cap H = \Phi$$

A regular  $T_1$ -space is called a  $T_3$ -space.

a) **Example 1:** Let  $X = \{a, b, c\}$  and let  $\tau = \{\Phi, \{a\}, \{b, c\}, X\}$ . Show that  $(X, \tau)$  is regular space but not a  $T_3$ -space.

b) **Solution:** The closed subsets are  $X, \{b, c\}, \{a\}, \Phi$ .

Consider the closed set  $\{b, c\}$  and the point  $a$  not belonging to it. Then  $\{b, c\}$  and  $\{a\}$  are open sets such that

$$\{b, c\} \subset \{b, c\}, a \in \{a\} \text{ and } \{b, c\} \cap \{a\} = \Phi$$

Similarly consider the closed set  $\{a\}$  and the point  $b$  not belonging to it. Then  $\{a\}$  and  $\{b, c\}$  are open sets such that

$$\{a\} \subset \{a\}, b \in \{b, c\} \text{ and } \{a\} \cap \{b, c\} = \Phi$$

Again for the closed set  $\{a\}$  and the point  $c$ , there exist open sets  $\{a\}$  and  $\{b, c\}$  such that

$$\{a\} \subset \{a\}, c \in \{b, c\} \text{ and } \{a\} \cap \{b, c\} = \Phi$$

It follows that  $(X, \tau)$  is regular space.

Since there does not exist a  $\tau$ -open set containing the point  $b$  and not containing the point  $c$ , the space is not  $T_1$  and consequently it is neither  $T_2$  nor  $T_3$ .

c) **Theorem:** A topological space  $X$  is regular iff for every point  $x \in X$  and every neighbourhood  $N$  of  $x$ , there exists a neighbourhood  $M$  of  $x$  such that  $\overline{M} \subset N$ . In other words, a topological space is regular iff the collection of all closed neighbourhoods of  $x$  forms a local base at  $x$ .

d) **Proof:** It is enough to prove the theorem for open neighbourhoods.

The only 'if' part: Let  $N$  be any nhd of  $x$ . Then there exists open set  $G$  such that  $x \in G \subset N$ . Since  $G'$  is closed and  $x \notin G'$ , by definition there exist open sets  $L$  and  $M$  such that  $G' \subset L, x \in M$  and  $L \cap M = \Phi$  so that  $M \subset L'$ .

It follows that 
$$M \subset (L') = L' \quad \dots(1)$$

Also 
$$G' \subset L \Rightarrow L' \subset G \subset N \quad \dots(2)$$

From (1) and (2), we get.  $M \subset N$ .

The only if part: Let the condition hold. Let  $F$  any closed set and let  $x \notin F$ . Then  $x \notin F'$ .

Since  $F'$  is open set containing  $x$ , by hypothesis there exists an open set  $M$  such that  $x \in M$  and  $\overline{M} \subset F'$ . Then  $(M')$  is an open set containing  $F$ . Also

$$M \cap M' = \Phi \Rightarrow M \cap (M') = \Phi$$

Hence the space is regular.

Another Statement: A topological space  $(X, \tau)$  is regular iff each open nhd of an element  $x \in X$  contains the closure of another open nhd of  $x$ .

**T4 - spaces or Normal Space**

2) **Definition :** A topological space  $(X, \tau)$  is said to be normal if and only if for every pair  $L, M$  of disjoint  $\tau$ -closed subsets of  $X$  there exists  $\tau$ -open sets  $G$  and  $H$  such that

$$L \subset G, M \subset H \text{ and } G \cap H = \Phi.$$

3) **Definition:** A normal  $T_1$ -space, is said to be a  $T_4$ -space.

Example : Let  $X = \{a, b, c\}$  and let  $\tau = \{\Phi, \{a\}, \{b, c\}, X\}$ . Show that (i)  $(X, \tau)$  is normal (ii) not  $T_2$  (iii) non-Hausdorff.

Solution: Hence the only pair of disjoint closed subsets of  $X$  is  $\{a\}, \{b, c\}$ . Also  $\{a\}, \{b, c\}$  are  $\tau$ -open sets such that

$$\{a\} \subset \{a\}, \{b, c\} \subset \{b, c\} \text{ and } \{a\} \cap \{b, c\} = \Phi$$

It follows that the space is normal. Again there does not exist a  $\tau$ -open set containing the point  $b$  and not containing point  $c$ .

Hence the space is not  $T_1$  and consequently it is neither  $T_2$  nor  $T_3$ . Thus the space is non- $T_3$  and non-Hausdorff space.

**II. CONCLUSION**

A property  $P$  of a topological space  $X$  is said to be weakly hereditary if and only if every closed subspace of  $X$  has the property  $P$ . Thus Lindelofness, normality and compactness are weakly hereditary properties.

### III. RESULT AND DISCUSSION

Theorem 1 : Every  $T_4$  -space is also a  $T_3$  -space.

Proof : Let  $(X, \tau)$  be a  $T_4$  -space. Then  $(X, \tau)$  is normal as well as  $T_1$ . To show that the space is  $T_3$ , it suffices to show that the space is regular. Let  $F$  be a  $\tau$  -closed subset of  $X$  and let  $x$  be a point of  $X$  such that  $x \notin F$ . Since  $(X, \tau)$  is a  $T_1$  -space,  $\{x\}$  is a closed subset of  $X$  such that  $\{x\} \cap F = \Phi$ . Then by normality, there open sets  $G$  and  $H$  such that  $\{x\} \subset G, F \subset H$  and  $G \cap H = \Phi$ . Also  $\{x\} \subset G \Rightarrow x \in G$ .

Thus there exist open sets  $G, H$  such that  $x \in G, F \subset H$  and  $G \cap H = \Phi$ . It follows that the space  $(X, \tau)$  is regular.

Theorem 2 : Every compact Hausdorff space is normal space ( $T_4$ ).

Proof : Let  $(X, \tau)$  be a compact Hausdorff space and let  $L, M$  be a pair of disjoint closed subsets of  $X$ . The space  $(X, \tau)$  is regular and so for each  $x$

$\in L$ , there exists  $\tau$ -open sets  $G_x$  and  $H_x$  such that  $x \in G_x, M \subset H_x$  and  $G_x \cap H_x = \Phi$ . Then the collection  $\{G_x : x \in L\}$  is an open cover of  $L$ .

$L$  is compact subspace of  $X$ . Hence there exists a finite number points,  $x_i, i = 1, 2, \dots, n$  in  $L$  such that

$L \subset \cup \{G_{x_i} : i = 1, 2, \dots, n\}$

Let  $G = \cup \{G_{x_i} : i = 1, 2, \dots, n\}$  and  $H = \cap \{H_{x_i} : i = 1, 2, \dots, n\}$

Then  $G, H$  are  $\tau$  -open sets such that

$L \subset G, M \subset H$  and  $G \cap H = \Phi$

Hence  $(X, \tau)$  is normal. Since every Hausdorff space is a  $T_1$ - space. It follows that  $(X, \tau)$  is also a  $T_4$  -space.

### REFERENCES

- [1] Schechter, Eric (1997). Handbook of Analysis and its Foundations. San Diego: Academic Press.
- [2] Willard, Stephen (1970). General topology. Reading, Mass.: Addison-Wesley Pub.
- [3] N. Levine, Generalized closed sets in topology, Rend. Ciri Math Palimo, 19(2) (1970), 89-96.
- [4] B.M. Munshi, separation axional Actaciencian India 12 (1986) 140-144.
- [5] Merrifield, Richard E.; Simmons, Howard E. (1989). Topological Methods in Chemistry. New York: Wiley.





10.22214/IJRASET



45.98



IMPACT FACTOR:  
7.129



IMPACT FACTOR:  
7.429



# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24\*7 Support on Whatsapp)