



iJRASET

International Journal For Research in
Applied Science and Engineering Technology



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 7 Issue: VII Month of publication: July 2019

DOI: <http://doi.org/10.22214/ijraset.2019.7068>

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Study to Linear Topological Space

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Abstract: In this thes several topics from Topology linear Algebra and Real Analysis are com-bined in the study of linear topological spaces. We begin with a brief look at linear spaces before moving on to study some basic properties of this structure of linear topological basis. Then we turn our attention to linear spaces with a metric topology. In particular we consider problems involving normed linear spaces bounded linear transformation and Hilbert spaces

Keywords: topology, linear algebra

I. INTRODUCTION

P. Thangavelu and Nithanantha Jothi introduced the concept of binary topology in (4). It is a single topological structure that carrier the subjects of a set x as well as the subsets of another set x for studying the information about the ordered pair $(A,B.)$ of subset of x and y . A linear topological space endowed with a topology such that the vector addition and sclar multiplication are both continuos the theory of linear topological spaces provide a remarkable economy in discussion of many classical mathematical problems. We introduce the concept of binary topology to linear section 2. We define the binary linear jtopology. Section 3 Space (BLTS) We prove that the binary product of two linear topological spcae is a BLTS. Also we discuss to concept of locally convese BLTS and locally bounded BLTS and prove some of their properties. In section 4 we define binary metric and binary normal. The main result of this section is that the binary product preserve metrizable and normbility. Section 5 deals with the construction of aBLTS using a family of binary seminorms.

II. PRELIMINARIES

1) *Definition:* Let x and y be any two non-empty and $d(x)$ and $g(y)$ be their power sets respectively. A binary topology from x to y is a binary structure $M \hat{=} d(x) \times d(y)$ that satisfies the following arioms (f, f) and $(x, y) \hat{=} M$

If (A_1, B_1) and $(A_2, B_2) \hat{=} M$, then $(A_1 \subset A_2, B_1 \subset B_2) \hat{=} M$.

If $\{(A_a, B_a) : a \in \hat{D}\}$ is a family of members of M ; then $(\cup_a \hat{D}A_a, \cup_a \hat{D}B_a) \hat{=} M$.

If M is a binary topology from x to y then the triplet (x, y, m) is called a binary topology space and the members of M are called binary points of binary open sets. (C,D) is called binary colsed if $(x | c, y | D)$ is binary open. The elements of x, x are called the binary points of the binary topological space (x, y, m) yet (x, y, m) be a binary topological space and let $(x, y) \hat{=} x \times x$ The binary open set (A, B) is called a binary neighborhood of (x, y) if $x \hat{=} A$ and $\hat{=} B$. If $x = y$ then M is called a binary topology on x and we write (x, M) as a binary space.

2) *Proposition:* Let (x, y, m) be a binary topological space. Then

(1) $T(M) = \{A \hat{=} x : (A, B) \hat{=} M \text{ for some } B \hat{=} Y\}$ is a topology on x .

$T1(M) = \{B \hat{=} Y : (A, B) \hat{=} M \text{ for some } A \hat{=} x\}$ is a topology on y .

III. BINARY LINEAR TOPOLOGY

1) *Definition:* A binary topology between two vector space is said to be binary linear if the two operation are continuous i. e, if V_1 , and V_2 are vector space over the some field k and for every neighbourhoods U of $(x_1 + x_2, y_1 + y_2) \hat{=} V_1 \times V_2$. ' two neighbourhoods U_1 and U_2 of (x_1, y_1) and (x_2, y_2) respectively that $U_1 + U_2 \hat{=} U$. Similarly for every neighbourhood W of $(lx, ly) v_1 \times v_2$ there exsits a neighbourhood w of (x, y) such that $lw \hat{=} w$. If M is a binary linary topology between two vector space V_1 and V_2 then triplet (V_1, V_2, M) is called a binary linear topological space (BLTS).

2) *Proposition:* If (V_1, T_1) and (V_2, T_2) are two linear topologica space then $(V_1, V_2, T_1 \times T_2)$ is a called the binary linear topological space.

- a) *Proof:* By proposition 2. 3, $(V_1, V_2, T_1 \times T_2)$ is a binary topological space. If remains to show that $T_2 \times T_2$ is a binary linear topology let $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ and (A_1, A_2) be a neighbourhood of $[(x_1, x_2) + (y_1, y_2)]$. Then $x_1 + y_1 \in A_1$ and $x_2 + y_2 \in A_2$. Since $A_1 \in T_1$ and $A_2 \in T_2$, and T_1 and T_2 are linear topological there exist neighbourhood B_1 and C_1 of x_1 and y_1 respectively in T_1 such that $B_1 + C_1 \in A_1$ and neighbourhood B_2 and C_2 of x_2 and y_2 respectively in T_2 such that $B_2 + C_2 \in A_2$. Then in $T_1 \times T_2$ (B_1, B_2) is a neighbourhood $(B_1, B_2) + (C_1, C_2) = T_1 \times T_2$ (B_1, B_2) Now Let (A_1, A_2) be a neighborhood of (x_1, x_2) in $T_1 \times T_2$ Then A_1 is a neighborhood of x_1 in T_1 and A_2 is a neighborhood of x_2 in T_2 . So there exists two B_1 and B_2 off x_1 and x_2 respectively such that $B_1 \in A_1$ and $B_2 \in A_2$. This implies that (B_1, B_2) is a neighbourhood of (x_1, x_2) such that $(B_1, B_2) \in (A_1, A_2)$. Thus $T_1 \times T_2$ is a binary linear topology.
- 3) *Proposition:* If (V_1, V_2, M) is a BLTS, then $a(M) = \{A \in V_1 : (A, B) \in M \text{ for some } B \in V_2\}$ is a linear topology on V_1 and $a(M) = \{B \in V_2 : (A, B) \in M \text{ for some } A \in V_1\}$ is a linear topology on V_2 .
- a) *Proof:* By proposition a (M) are both topologies in V_1 and V_2 respectively. Let $x_1, y_1 \in V_1$ and $A \in a(M)$ contains $x_1 + y_1$. Then for some $x_2, y_2 \in V_2$ there exists $B \in V_2$ such that $(x_1 + y_1, x_2 + y_2) \in (A, B)$ Where $(A, B) \in M$, since M is a binary linear topology, there exist (E_1, E_2) and (F_1, F_2) in M such that $(x_1, x_2) \in (E_1, E_2), (y_1, y_2) \in (F_1, F_2)$ and $(E_1, E_2) + (F_1, F_2) \in (A, B)$. $x_1 \in E_1, y_1 \in F_1$ and (E_1, E_2) by the definition of binary sets. Also E_1 and $F_1 \in a(M)$ by the construction of (T). Similarly for $x_2, y_2 \in V_2$. Where $A \in a(M)$ we can find also a linear of x say U such that $x \in U$. Thus $a(M)$ is linear topology in the same way we can prove that (M) topology.
- 4) *Definition:* A local base of a binary linear topology (V_1, V_2, M) is the base Consists of the neighborhood of a binary points (x, y)
- 5) *Definition:* A set $(A, B) \in d(V_1) \times d(V_2)$ is convex if for all pairs $(x_1, x_2), (y_1, y_2) \in (A, B)$ $(1 - \lambda)(x_1, x_2) + \lambda(y_1, y_2) \in (A, B)$ $\lambda \in (0, 1)$.
- 6) *Definition:* A binary topology is called locally convex if there exist a local base at $(0, 0)$ whose members are convex.
- 7) *Definition:* A BLTS is locally bounded of $(0, 0)$ as a bounded neighbourhood, i.e., a neighbourhood (E, F) such that $(N, M) \in No$. the set of neighbourhood of $(0, 0)$ there exists $S \in R$ such that $t \in S, (E, F) \in t(N, M)$. Let (V_1, V_2, M) be a BLTS. Then for every $(w_1, w_2) \in No$. 'balanced and symmetric sets $(x_1, y_1), (x_2, y_2) \in No$ such that $(x_1, x_2) \in t(x_2, y_2) \subset (w_1, w_2)$.
- a) *Proof:* If $(w_1, w_2) \in No$ then w_1 and w_2 are neighbourhood of O in $(V_1, T(M))$ and (V_2, T) respectively by the property of linear topologies there exists symmetric balanced neighbourhood of $0, x_1, x_2 \in T(M)$ and $y_1 + y_2 \in CW_2$ Now, $x_1, y_1 \in P \in R$ with $|a| \leq 1, a x_1 \in x_1$ and $y_1 \in y_1$.
- So $a(x_1, y_1) = (a x_1, y_1) \subset (x_1, y_1)$ thus (x_1, y_1) and (x_2, y_2) are balanced by the symmetry of x_1 and y_1 we get $x_1 = -x_1, y_1 = -y_1 \in P(x_1, y_1) = (-x_1, y_1)$ thus (x_1, y_1) is symmetric and similarly (x_2, y_2) is also symmetric. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \subset (w_1, w_2)$.
- 8) *Proposition:* Let V_1 and V_2 be real vector space and U_1 be a convex set in V_1 and U_2 be a convex set in V_2 then (U_1, U_2) is convex $d(V_1) \times d(V_2)$.
- a) *Proof:* Let $(x_1, y_1) \in (U_1, U_2)$ for $i = 1, 2$ then $x_1 \in U_1, y_1 \in U_2$ for $i = 1, 2$ $\lambda x_1 + (1 - \lambda) x_2 \in U_1$ for $0 \leq \lambda \leq 1$. So $(\lambda x_1 + (1 - \lambda) x_2, y_1) \in (U_1, U_2)$. Consider $(\lambda x_1, y_1) + (1 - \lambda)(x_2, y_2) = (\lambda x_1, y_1) + (1 - \lambda)(x_2, y_2) \in (U_1, U_2)$ for $\lambda \in [0, 1]$. Thus (U_1, U_2) is convex.
- 9) *Corollary:* If (V_1, T_1) and (V_2, T_2) are both locally convex topological vector spaces then their binary product $(V_1, V_2, T_1 \times T_2)$ is locally convex BLTS.
- 10) *Proposition:* Let U_1 and U_2 be bounded sets in two real vector spaces V_1 and V_2 respectively then bounded.
- a) *Proof:* Since U_1 is bounded for every neighbourhood $\epsilon \in No$. (V_1) . 's1 $\in R$ such that $t \in E_2 \in No$. (V_2) , 's2 $\in R$ such that $t > S_1, V_1 \subset tE_1$. Similarly for every neighbourhood $E_2 \in No$. (V_2) , 's2 $\in R$ such that $t > S_2, U_2 \subset tE_2$. Let $T_1 \in R$ correspond to E and $T_2 \in R$ to f then $t > t_1, U_1 \subset tE$ and $t > t_2, U_2 \subset tf$. So $t > S$, where $S = \max(t_1, t_2), U_1 \subset tE$ and $U_2 \subset tf$ i.e. $(U_1, U_2) \in t(E, F), t > S$. Thus (U_1, U_2) is bounded.
- 11) *Corollary:* If (V_1, T_1) and (V_2, T_2) are both locally bounded topological vector spaces, then their binary product $(V_1, V_2, T_1 \times T_2)$ is a locally bounded BLTS.
- 12) *Proposition:* Let (V_1, T_1) be a topological vector space and V_2 be another vector space such that map $T : V_1 \otimes V_2$ is an isomorphism. Then $T_2 = \{T(A) : A \in T_1\}$ is a linear topology in V_2 and hence $T_1 \times T_2$ is a binary linear topology from V_1 to V_2 .
- a) *Proof:* Since T is an isomorphism, $T(f) = f$ and $T(V_1) = V_2$ and So $f \in V_2$ and So $f \in V_2 \in T_2$. Let $A, B \in T_2$. Then $A = T(A')$ and $B = T(B')$ for some A' and $B' \in T_1$. So $A' \subset B' \in T_1$ $(A' \subset B') \in T_2$ $T(A' \subset B') = T(A') \subset T(B') = A \subset B$ Thus. $A \subset B \in T_2$. Now Let $\{A_\alpha\} \dots T_2$ for some index set. T then exists $(B_\alpha) \dots T_1$ Such that $A_\alpha = T(B_\alpha)$ for each $\alpha \in T$ Then $U \dots T \in T_2$ for each $\alpha \in T$. So $x_1 + y_1 \in A$ and $U \dots T A_\alpha = U \dots T T(B_\alpha)$. Then $B_1, B_2 \in T_2$ and $x_1 \in A_1 \in P x_2 = T(x_1) \in T(A_1) = B_1 y_1$.

IV. BINARY MERITABLE AND BINARY NORMABLE BLTS

1) *Definition:* A binary metric on two sets V_1 and V_2 is a map $d : (V_1 \times V_2) \times (V_1 \times V_2) \rightarrow \mathbb{R}$ satisfying the following axioms : If $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ then.

$d[(x_1, x_2), (y_1, y_2)] \geq 0$ and $d[(x_1, x_2), (y_1, y_2)] = d[(y_1, y_2), (x_1, x_2)]$ and.

$d[(x_1, x_2), (y_1, y_2)] \leq d[(x_1, x_2), (z_1, z_2)] + d[(z_1, z_2), (y_1, y_2)]$ for every $(z_1, z_2) \in V_1 \times V_2$

$d[(x_1, x_2), (y_1, y_2)] = 0 \iff x_1 = x_2$ and $y_1 = y_2$.

2) *Definition:* Let (V_1, V_2, M) be a BLTS. A binary topology M is metrizable with a binary metric d if for any (x, y) in some binary open set $(A, B) \in M, \forall \epsilon > 0$ Such that $B_\epsilon(x, y)$

is contained in (A, B) where π_i is the projection map to V_i for $i = 1, 2$.

3) *Proposition:* If (V_1, T_1) and (V_2, T_2) are two linear topological space such that T_1 and T_2 are both metrizable with metrics d_1 and d_2 respectively then $T_1 \times T_2$ are both metrizable with metrics d_1 and d_2 respectively then $T_1 \times T_2$ is binary metrizable.

a) *Proof:* Consider the map $d : (V_1 \times V_2) \times (V_1 \times V_2) \rightarrow \mathbb{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2}, (x_1, x_2), (y_1, y_2) \in (V_1 \times V_2)$$

If $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ then

$$(1) \quad d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2} \geq 0, \text{ since } d_1, d_2 \geq 0$$

$d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ are both non-negative.

$$(2) \quad d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2} = 0 \iff d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0$$

This happens if and only if $x_1 = y_1$ and $x_2 = y_2$ i. e. when $(x_1, x_2) = (y_1, y_2)$

$$(3) \quad d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2} = \frac{d_1(y_1, x_1) + d_2(y_2, x_2)}{2}$$

$d[(x_1, x_2), (y_1, y_2)] = d[(y_1, y_2), (x_1, x_2)]$ and if $(z_1, z_2) \in V_1 \times V_2$

$$d[(x_1, x_2), (z_1, z_2)] \leq d_1(x_1, z_1) + d_2(x_2, z_2)$$

$$(4) \quad d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2} = \frac{d_1(x_1, z_1) + d_2(x_2, z_2)}{2} + \frac{d_1(z_1, y_1) + d_2(z_2, y_2)}{2}$$

$$d_2(x_2, y_2) = d[(x_1, x_2), (z_1, z_2)] + d[(z_1, z_2), (y_1, y_2)].$$

Thus d is a binary metric let $(A, B) \in T_1 \times T_2$ and $(x, y) \in (A, B)$ Then $x \in A$ and $y \in B$ since T_1 and T_2 are metrizable. $\forall \epsilon_1, \epsilon_2 > 0$ with respect to d_1 and d_2 respectively such that $B_{\epsilon_1}(x) \subset A$ and $B_{\epsilon_2}(y) \subset B$. i. e. if $d_1(x_1, x_2) < \epsilon_1$ then $x_1 \in B_{\epsilon_1}(x)$ and if $d_2(y_1, y_2) < \epsilon_2$ then $y_1 \in B_{\epsilon_2}(y)$. Let $(x, y) \in (A, B)$ let $r = \min(\epsilon_1, \epsilon_2)$ and $(u, v) \in B_{r/2}(x, y)$ then $(x, y), (u, v) \in B_{r/2}(x, y)$, i. e. $d_1(x_1, u_1) + d_2(y_2, v_2) < r/2$. So $d_1(x_1, u_1) + d_2(y_2, v_2) < r/2$, i. e. $d_1(x_1, u_1) + d_2(y_2, v_2) < r/2$. So $d_1(x_1, u_1) + d_2(y_2, v_2) < r/2$ and $d(y, v) < r/2$ Hence $u \in B_r(x)$

$(x, y) \in (A, B)$ and $(u, v) \in B_{r/2}(x, y) \subset (A, B)$ showing that $B_{r/2}(x, y) \subset (A, B)$.

4) **Definition:** A binary seminorm on two vector space V_1 and V_2 is a map $\|\cdot\| : V_1 \times V_2 \rightarrow \mathbb{R}$ such that for each $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$.

$$\|(x_1, x_2)\| \geq 0$$

$$\|a(x_1, x_2)\| = |a| \|(x_1, x_2)\|$$

$$\|(x_1, x_2) + (y_1, y_2)\| \leq \|(x_1, x_2)\| + \|(y_1, y_2)\|$$

A binary seminorm becomes a binary norm if the following condition holds.

$$\|(x_1, x_2)\| = 0 \iff (x_1, x_2) = (0, 0)$$

5) **Proposition:** If (V_1, T_1) and (V_2, T_2) are both normable topological vector space, then their binary product is binary normable.

a) **Proof:** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the norms corresponding to t_1 and t_2 respectively. Then we get two metrics d_1 and d_2 defined by $d_1((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\|_1$, $i = 1, 2$ and $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ with which t_1 and t_2 are metrizable respectively. So by proposition $T_1 \times T_2$ is metrizable with which T_1 and $d_1(x_1, y_1) + d_2(x_2, y_2) \in V(x_1, x_2), (y_1, y_2) \in (V_1 \times V_2)$. Hence the binary norm $\|\cdot\|$ defined by $\|(x_1, x_2)\|$ corresponds to the topology $T_1 \times T_2$ but

$$\text{this norm is same as } \frac{\|\cdot\|_1 + \|\cdot\|_2}{2} \quad \text{since } \|(x_1, x_2)\| = d(x_1, x_2, (0, 0)) = \frac{d_1(x_1, 0) + d_2(x_2, 0)}{2}$$

$$= \frac{\|x_1 - 0\|_1 + \|x_2 - 0\|_2}{2} = \frac{\|x_1\|_1 + \|x_2\|_2}{2}$$

6) **Lemma:** Let V_1 and V_2 be two vector space and P be a binary seminorm on $V_1 \times V_2$

Then there exists two seminorm P_1 and P_2 on V_1 and V_2 respectively.

a) **Proof:** Let $P_1 : V_1 \rightarrow \mathbb{R}$ be defined by $P_1(x) = \inf\{P(x, y) : y \in V_2\}$ since $P(x, y) \geq 0, (x, y) \in V_1 \times V_2, P_1(x) \geq 0 \forall x \in V_1$ and $\alpha \in \mathbb{R}, P_1(\alpha x) = \inf\{P(\alpha x, y) : y \in V_2\}$

$$= \inf\{|\alpha| P(x, y) : y \in V_2\}$$

$$= |\alpha| \inf\{P(x, y) : y \in V_2\}$$

$$= |\alpha| P_1(x)$$

$$\text{for } x, y \in V_1, P_1(x+y) = \inf\{P(x+y, z) : z \in V_2\}$$

$$= \inf\{P(x+y, z_1+z_2) : z_1, z_2 \in V_2\}$$

$$z = z_1 + z_2$$

$$= \inf_{z_1, z_2} \{P(x, z_1) + P(y, z_2) : z_1, z_2 \in V_2\}$$

$$z_1, z_2$$

$$= \inf_{z_1, z_2} \{P(x, z_1) + P(y, z_2) : z_1, z_2 \in V_2\}$$

$$z_1, z_2$$

Thus $P_1(x+y) \leq P_1(x) + P_1(y)$

Hence P_1 is a seminorm on V_1 and family $P_2 : V_2 \rightarrow \mathbb{R}$ defined by $P_2(y) = \inf\{P(x, y) : x \in V_1\}$ is a seminorm on V_2 .

V. CONCLUSION

In This paper we have introduced the concept of linear topological space to situation in which we have to deal with two vector space and a topology between the spaces. This helps to study both the space simultaneously. The concept of topological vector space is well used in mathematics engineering and science and particularly is quantum mechanics. Hence our theory of Binary linear Topological space helps in the future development of such areas.

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