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# Matrix Games with Uncertain Entries

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**Abstract:** *In classical game theory, a conflict of two opponents can be modelled as an equilibrium-based matrix game. We assume a conflict of two non-cooperative antagonistic opponents with a finite number of strategies with zero-sum or constant sum pay-offs. At the same time, we suppose that the elements of the payoff matrix describing the game are not fixed and are allowed to change within a specified interval.*

*Supposing that some of the elements of the payoff matrix are uncertain, it is evident that this would influence the utilities of both players at the same time and moreover, such entropy of the model would eventually influence the position of equilibria or its very existence. We propose a modelling approach that allows one to find a solution of the game with either pure or mixed strategies of opponents with the guaranteed payoffs under the assumption that a specified number of unspecified entries would attain different values than expected. The chosen robust approach is presented briefly as well as the necessary circumstances of matrix game solutions. Our novelty approach follows and is accompanied by an explanatory example in the end of the paper.*

**Keywords:** *Game theory, matrix games, robust approach, equilibria, uncertainty*

## I. INTRODUCTION

Decision-making process is heavily influenced by the uncertainty in the decision-making situation and the inaccuracy of the data. From this perspective, it is possible to analyze the results of models of game theory, if we assume possible changes in payoffs. Payoff changes affect the choice of the players' best strategies (pure or mixed) and moreover, a change that is advantageous to one of the players may be disadvantageous to the other. There are different approaches for treating uncertainty in operations research. A stochastic approach is probably the one that comes to one's mind when uncertainty is mentioned. This approach is probabilistic-based. A fuzzy approach is also available when it comes to calculations with inexact parameters. We want to take advantage of robust optimization tools that are strictly deterministic and rather set-based than probabilistic to deal with the goals of this paper. Since the matrix game can be reformulated as a linear optimization problem, we are going to treat the uncertainty here in a sense of robust linear optimization. The parameter uncertainty in the linear optimization has been dealt with in many ways since the pioneering work of Dantzig that discusses the very matter of linear programming under uncertainty. The first extensive work on the matters of robust optimization is considered by Soyster. In his paper, he proposes the reformulation of a linear program into the new program that contains the aspects of uncertainty and is able to provide robust solutions that are "protected" against the adverse changes in parameters. The Soyster's approach preserves linearity, it tends to be, however, very over-conservative meaning that it seeks to protect the model results at all costs from all possible uncertainty influences which also brings a significant losses in objective function value. Much later, Ben-Tal and Nemirovski proposed an approach where the conservativeness of the protection can be controlled by additional parameters, while this formulation leads to a conic quadratic programming that is non-linear itself. Later on, a novelty approach was introduced by Bertsimas and Sim that allows controlling the level of conservatism in the solution in a way that leads once again to a linear optimization model that can even be applied to discrete optimization problems. This approach was further extended by Büsing and D'Andreagiovanni and subsequently elaborated in detail in 2014 by the same authors (Büsing and D'Andreagiovanni). Finally, it is worth to mention an extensive work of Ben-Tal, El Ghaoui and Nemirovski describing the issues of Robust optimization as a whole under the different perspectives. The following paragraph discusses the results achieved on uncertainty in game theory by other authors. The probabilistic approach is presented by Pun and Wong wherein it is shown how one can possibly apply their approach specifically in the insurance, which is modelled as a game here. Fuzzy approach is also very popular in game theory. Namely in matrix-type of games, the pareto-optimal solutions with fuzzy payoffs are discussed by Aggarwal, Chandra and Mehra or in a bit more specific fashion by Dutta and Gupta who discuss the Nash equilibrium in the games with trapezoidal fuzzy payoffs. Similar matters are presented for bi-matrix games under the fuzzy entries by Roy and Mula while Li is concerned with the triangular fuzzy entries and also an effective approach to computation of the problem. It is discussed by the same author, how one is able to treat the matrix games using deterministic approach, specifically the interval arithmetic while he also demonstrates the means of reaching the results of the game by linear programming. While in Li's approach the outcome of the game is once again interval data, we propose an approach seeking robust solutions such that input parameters are considered to deviate within an assumed interval.

## II. MATERIALS AND METHODS

In this section, a way of transformation of a generic linear optimization model to its robust counterpart will be described, followed by mathematical description of a general matrix game and its reformulation as a linear program.

### A. Robust Programming Introduction

In the following paragraph, it is described how a linear optimization model can be transformed into its robust counterpart, that is, a modified program that seeks robust optimal solution. Such solution is resistant against any deviations that might occur anywhere in the original deterministic model. For the description we utilize the approach of “Γ-robustness” of also described in our previous paper. We assume a generic linear optimization model:

$$\begin{aligned} & \max \sum_{j=1}^n c_j x_j \\ & \text{S.t.} \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, i = 1, \dots, m \\ & x_j \geq 0, j = 1, \dots, n \end{aligned} \tag{1}$$

When the presence of uncertainty is considered, we usually assume that the problem coefficients  $a_{ij}, b_i, c_j$  (or at least some of them) are not precisely defined. For purposes of this article, let us consider the uncertainty to be present within the set of coefficients  $a_{ij}$  from now on. This leads to a new problem:

$$\begin{aligned} & \max \sum_{j=1}^n c_j x_j \\ & \text{S.t.} \\ & \sum_{j=1}^n (a_{ij} + \delta_{ij}^a) x_j \leq b_i, i = 1, \dots, m \\ & x_j \geq 0, j = 1, \dots, n \end{aligned} \tag{2}$$

The (2) represents the reformulation of (1) with uncertain coefficients  $a_{ij}$ . The uncertainty is expressed using deviations for any coefficients if need be. We assume any deviation  $\delta_{ij}^a$  to be any real nonzero number. It was illustrated by Ben-Tal, El Ghaoui and Nemirovski that even a slight change in the original coefficient value may affect the optimal solution adversely. In some cases, it may even become infeasible while this is caused by a small deviation. Before the construction of the robust counterpart of (1), Bertsimas and Sim propose 4 assumptions that must hold:

- 1) For each coefficient  $a_{ij}$ , one is able to define its deterministic (expected, usual,...) value and its maximum deviation  $\delta_{ij}^a$  from the deterministic value.
- 2) The deterministic value  $a_{ij}$  then belongs to the symmetric interval  $[a_{ij} - \delta_{ij}^a, a_{ij} + \delta_{ij}^a]$
- 3) The uncertain coefficients are stochastically independent random coefficients, each one with its own deviation range
- 4) For each constraint  $i$ , one is able to define a maximum number of coefficients  $\Gamma_i$ , that will deviate simultaneously from its deterministic value in the constraint  $i$ .

Then, a robust model can be constructed, based upon deviations and it is possible to reach the final form of this robust counterpart of (1):

$$\begin{aligned} & \max \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j + \Gamma_i z_i + \sum_{j \in Q_i} q_{ij} \leq b_i, i = 1, \dots, m \\ & z_i + q_{ij} \geq \delta_{ij}^a x_j, i = 1, \dots, m, \forall j \in Q_i \\ & z_i \geq 0, i = 1, \dots, m \\ & q_{ij} \geq 0, i = 1, \dots, m, \forall j \in Q_i \\ & x_j \geq 0, j = 1, \dots, n \end{aligned} \tag{3}$$

where the parameter  $\Gamma_i, 0 \leq \Gamma_i \leq |Q_i|, i = 1, \dots, m$ , controls the protection against uncertainty in the constraint  $i, q_{ij}$  is an auxiliary variable for each  $a_{ij}$  that is considered uncertain,  $z_i$  is another auxiliary variable merely preserving a relationship between the first and second constraint and  $Q_i$  indicates a set of indices  $j$  of those  $a_{ij}$  for which the deviation is actually considered. The creation of (3) is based on exploiting the properties of primal and dual versions of the original problem (1).

### B. Matrix Game

Let us consider the basic type of a game called matrix game or normal-form game. This game describes a zero sum conflict of two players  $H_1$  and  $H_2$  each one with the finite number of strategies  $R_i, i \in \{1, \dots, m\}$  and  $S_j, j \in \{1, \dots, n\}$  respectively. A matrix  $A \in \mathbb{R}^{m \times n}$  with entries  $a_{ij}$  is called payoff matrix. All entries represent payments from  $H_2$  and  $H_1$  and since we deal with the zero-sum game, it is possible to construct an equivalent payoff matrix for opposite direction payments  $\underline{A} \in \mathbb{R}^{m \times n}$  with entries  $\underline{a}_{ij} = -a_{ij} \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ .

The game and its solution is based on von-Neumann's minimax theorem meaning that both players seek to minimize the maximum payoff of the other player. Thus, it is necessary to find vectors  $x^*$  and  $y^*$  of optimal strategies portfolio for both players respectively. Componentwise  $r^* = (r_1, r_2, \dots, r_m)$  and  $s^* = (s_1, s_2, \dots, s_n)$  show the optimal usage of any strategy  $R_i(S_j)$  for both players. Necessarily  $\|r^*\|_1 = 1$  and  $\|s^*\|_1 = 1$  for any case. We seek such pair of vectors  $r^*, s^*$  that  $\min_s r^T A s = \max_r (s) \max(r) r^T A s$  in the game. One is able to determine the solution easily in a sense of pure strategies. This can be achieved by applying the minmax (or maxmin) approach directly in the matrix to determine which particular strategy (or strategies in case there is more equilibria in the game) optimize the payoffs. In general, there does not have to be any equilibrium such that usage of a unique strategy optimizes the payoffs of both players. In that case, it is necessary to determine all components of  $r^*, s^*$  to obtain frequencies with which the strategies should be used to maximize outcomes while minimizing losses.

Any matrix game can be reformulated as a linear program based on the minimax assumption while normalizing the optimal strategy vectors to 1. The minimax objective formulation is, however, not a linear expression and it is generally NP-hard in terms of solution complexity. By introducing a new variable  $w$  as the first player's outcome, it is possible to bound the expected payoff of the player as  $r^T a_j \geq w, w = (w, \dots, w), \forall j \in \{1, \dots, n\}$  while we maximize  $w$ . Similarly, this can be done for the second player by bounding the objective from above by a variable  $w$  in the following way:  $a_i s \leq w, \forall i \in \{1, \dots, m\}$  while minimizing  $w$ . Considering the both substitutions, it is possible to formulate a pair of linear programs

$$\begin{array}{ll}
 \max w & \min w \\
 \text{s.t.} & \text{s.t.} \\
 r^T a_j \geq w, j = 1, \dots, n & a_i s \leq w, i = 1, \dots, m \\
 \sum_{i=1}^m r_i = 1 & \sum_{j=1}^n s_j = 1 \\
 x \geq 0 & y \geq 0
 \end{array} \tag{4}$$

These programs can be transformed into a pair of dual linear programs.

$$\begin{array}{ll}
 w^* = \min \sum_{i=1}^m x_i & w^* = \max \sum_{i=1}^m y_i \\
 \text{s.t.} & \text{S.t.} \\
 x^T \hat{a}_j \geq 1, j = 1, \dots, n & \hat{a}_i y \geq 1, j = 1, \dots, m \\
 x \geq 0 & y \geq 0
 \end{array} \tag{5}$$

From the strong duality theorem, for these particular programs it holds that the optimal solutions are equal. This value lies between upper and lower value of the matrix game  $\underline{w} \leq w^* \leq \bar{w}$ .

Finally, before the linear programs can be practically solved, one must deal with the fact that the both models will only yield desired results when all entries  $a_{ij}$  are positive.

The entries in the real life situations can also be non-positive. This can be easily dealt with by transforming all entries additively by some  $N$  that will shift all values positive in the following fashion:  $\alpha = \left| \min_{ij} a_{ij} \right| + c$ , with a positive constant  $c$  and new entries  $\hat{a}_{ij}$  are  $\hat{a}_{ij} = a_{ij} + \alpha, \forall i, j$ .

For a matrix game, the following theorem is true, which shows that if players play optimal strategies, both will achieve the best possible result.



C. *The Minimax Theorem.*

For every finite two-person zero-sum game.

- 1) There is a number  $w^*$ ,  $\underline{w} \leq w^* \leq \bar{w}$  called the value of the game,
- 2) There is a mixed strategy for Player  $H_1$  such that his average gain is at least  $w^*$  no matter what  $H_2$  does, and
- 3) There is a mixed strategy for Player  $H_2$  such that his average loss is at most  $w^*$  no matter what  $H_1$  does.

III. RESULTS

The assumption that some of the input data of the matrix game are not precisely known, it leads a decision maker to finding robust solution. The concept of building a robust counterpart was briefly described in chapter 2.1 and it allows a player to embed uncertainties into the left-hand side matrix of the system  $\hat{A}x \geq 1$  or  $y^T \hat{A} \leq 1$ . Considering the dual linear programs defining the game from both views, the only changeable parameters lie within the left-hand side coefficients. This also make sense in terms of the original game representation by the payoff table where the only input data were actually the payoffs  $a_{ij}$ .

A. *Robust Formulation of the Matrix Game*

We will take advantage of the formulation of robust counterpart in (3) and specify it for the particular problems of the matrix game. Note that the indices  $i, j$  represent rows and columns of  $A_{ij}$  whereas in the following model the representation is transposed:

$$\begin{aligned}
 \underline{w} = \min \sum_{i=1}^m x_i & \qquad \qquad \qquad \underline{w} = \max \sum_{i=1}^m y_i \\
 \text{s.t.} & \qquad \qquad \qquad \text{S.t.} \\
 \sum_{i=1}^m x_i \hat{a}_{ij} - \Gamma_j z_j - \sum_{i \in Q_j} q_{ji} \geq 1, j = 1, \dots, n & \qquad \sum_{i=1}^m y_i \hat{a}_{ij} - \Gamma_j z_j - \sum_{i \in Q_j} q_{ji} \leq 1, i = 1, \dots, m \quad (6) \\
 z_j + q_{ji} \geq \delta \frac{a_{ji}}{j_i} x_i, \forall i \in Q_j, j = 1, \dots, n & \qquad z_i + q_{ji} \geq \delta \frac{a_{ji}}{j_i} y_i, \forall i \in Q_j, i = 1, \dots, m \\
 q_{ji} \geq 0, \forall i \in Q_j, j = 1, \dots, n & \qquad q_{ji} \geq 0, \forall i \in Q_j, j = 1, \dots, m \\
 x_i \geq 0, i = 1, \dots, m & \qquad x_i \geq 0, j = 1, \dots, n
 \end{aligned}$$

where the parameter  $\Gamma_j, 0 \leq \Gamma_j \leq |Q_j|$  controls the protection against uncertainty in a chosen constraint, i.e. how many coefficients  $a_{ji}$  are expected to deviate from its deterministic value in the constraint  $j, i$  resp.

Because these models do not create the pair of dual linear programs, it is possible to suppose, that  $\underline{w} \leq \bar{w}$  which means that the worst outcome of the first player according to the first player selection of strategy is lower than the first player's outcome according to the second player's selection of strategy.

B. *Practical Example*

To illustrate the usage of the proposed robust approach, we will demonstrate it on the following small-scale matrix game with an arbitrarily chosen input. Two players  $H_1, H_2$  have available their strategy sets  $R = \{R_1, R_2, R_3\}$  and  $S = \{S_1, S_2, S_3, S_4\}$  with the payoff matrix  $A$  and the maximum possible deviations of some of payoffs in matrix  $X$  chosen arbitrarily as follows:

$$\begin{aligned}
 A &= \begin{matrix} & 1 & 2 & 5 & 7 \\ 1 & 4 & 2 & 1 & 8 \\ & 2 & 5 & 3 & 6 \\ & 0 & 0 & 0 & 0 \end{matrix} \\
 \Delta &= \begin{matrix} & 3 & 0 & 0 & 4 \\ & 1 & 1 & 2 & 5 \end{matrix}
 \end{aligned}$$

Similarly, the optimal strategy for the second player is  $s^* = (0.5; 0; 0.5; 0)$  and the value of the game is  $w = 4$ . And again, this outcome is better than the outcome that can be obtained in case of increasing of all payoffs (respective upper value of the game is  $\bar{w} = 5$ ).

Analysis of a matrix game with uncertain evaluation of payoffs can be done using the robust approach. The value received by solving the robust model for the first player can be seen as the lower bound of outcomes of the players, which is higher than the lower value of the game with the lowest payoffs. In contrary, the value gained by solving the robust model for the second player is the upper bound of the players' outcomes, which is lower than the upper value of the game with the highest payoffs. Of course, these bounds depend on the assumptions of the supposed changes of payoffs. Advantage of this way of game analysis is that we receive the tighter interval of possible outcomes.

#### IV. CONCLUSION

We have explored the possibilities of applying  $\Gamma$ -robustness approach in the matrix games of two players. It was discussed in the introduction part that the different techniques could be used for evaluation when uncertain entries of the payoff matrix are present. Each technique brings the decision maker slightly different point of view on the results achieved by computations, i.e. probabilistic approach leads to probabilistic outcomes, fuzzy entries will lead to fuzzy character of the result etc. We applied an approach that is solely deterministic and generates deterministic results, too. Using this approach not only allows the decision maker to control a level of protection against uncertainty by the  $\Gamma$  parameter but it also occurred, as shown in our example, that the evaluation by robust approach leads to tighter bounds on the value of the game. It remains to our future work to explore how does this particular robust approach generally influences an existence of equilibria of the game.

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