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# On the Positive Pell Equation

$$y^2 = 8x^2 + 49$$

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**Abstract:** The binary quadratic Diophantine equation represented by the positive Pellian  $y^2 = 8x^2 + 49$  is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

**Keywords:** Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

## I. INTRODUCTION

The binary quadratic equations of the form  $y^2 = Dx^2 + 1$  where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by  $y^2 = 8x^2 + 49$  is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

## II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 8x^2 + 49 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 2, y_0 = 9 \quad D = 8$$

The Pellian equation is

$$y^2 = 8x^2 + 1 \tag{2}$$

The initial solution of Pellian equation is

$$\tilde{x}_0 = 1, \tilde{y}_0 = 3,$$

The general solution  $(\tilde{x}_n, \tilde{y}_n)$  of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{8}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1}$$

$$g_n = (3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1}$$

Applying Brahmagupta lemma between  $(x_0, y_0)$  and  $(\tilde{x}_n, \tilde{y}_n)$  the other integer solution of (1) are given by,

$$x_{n+1} = f_n + \frac{9}{2\sqrt{8}} g_n$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{8}{\sqrt{8}} g_n$$

The recurrence relation satisfied by the solution  $x$  and  $y$  are given by,

$$x_{n+3} - 6x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 6y_{n+2} + y_{n+1} = 0$$

Some numerical examples of  $x$  and  $y$  satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	$x_n$	$y_n$
0	2	9
1	15	43
2	88	249
3	513	1451
4	2990	8457

From the above table, we observe some interesting relations among the solutions which are presented below.

- A. Both  $x_n$  and  $y_n$  values are odd and even.
- B. Each of the following expression is a nasty number:

1)  $\frac{6}{49} [98 + 18y_{2n+2} - 32x_{2n+2}]$

2)  $\frac{6}{49} [98 + 18x_{2n+3} - 86x_{2n+2}]$

3)  $\frac{6}{147} [294 + 9x_{2n+4} - 249x_{2n+2}]$

4)  $\frac{6}{147} [294 + 18y_{2n+3} - 240x_{2n+2}]$

5)  $\frac{6}{833} [1666 + 18y_{2n+4} - 1408x_{2n+2}]$

6)  $\frac{6}{147} [294 + 86y_{2n+2} - 32x_{2n+3}]$

7)  $\frac{6}{833} [1666 + 498y_{2n+2} - 32x_{2n+4}]$

8)  $\frac{6}{392} [784 + 240y_{2n+2} - 32y_{2n+3}]$

9)  $\frac{6}{2352} [4704 + 1408y_{2n+2} - 32y_{2n+4}]$

10)  $\frac{6}{49} [98 + 86y_{2n+3} - 240x_{2n+3}]$

11)  $\frac{6}{49} [98 + 86x_{2n+4} - 498x_{2n+3}]$

12)  $\frac{6}{147} [294 + 86y_{2n+4} - 1408x_{2n+3}]$

$$13) \frac{6}{147} [294 + 498y_{2n+3} - 240x_{2n+4}]$$

$$14) \frac{6}{49} [98 + 498y_{2n+4} - 1408x_{2n+4}]$$

$$15) \frac{6}{392} [784 + 1408y_{2n+3} - 240y_{2n+4}]$$

C. Each of the following expressions is a cubical integer.

$$1) \frac{1}{49} [18y_{3n+3} - 32x_{3n+3} + 54y_{n+1} - 96x_{n+1}]$$

$$2) \frac{1}{49} [18x_{3n+4} - 86x_{3n+3} + 54x_{n+2} - 258x_{n+1}]$$

$$3) \frac{1}{147} [9x_{3n+5} - 249x_{3n+3} + 27x_{n+3} - 747x_{n+1}]$$

$$4) \frac{1}{147} [18y_{3n+4} - 240x_{3n+3} + 54y_{n+2} - 720x_{n+1}]$$

$$5) \frac{1}{833} [18y_{3n+5} - 1408x_{3n+3} + 54y_{n+3} - 4224x_{n+1}]$$

$$6) \frac{1}{147} [86y_{3n+3} - 32x_{3n+4} + 258y_{n+1} - 96x_{n+2}]$$

$$7) \frac{1}{833} [498y_{3n+3} - 32x_{3n+5} + 1494y_{n+1} - 96x_{n+3}]$$

$$8) \frac{1}{392} [240y_{3n+3} - 32y_{3n+4} + 720y_{n+1} - 96y_{n+2}]$$

$$9) \frac{1}{2352} [1408y_{3n+3} - 32y_{3n+5} + 4224y_{n+1} - 96y_{n+3}]$$

$$10) \frac{1}{49} [86x_{3n+5} - 498x_{3n+4} + 258x_{n+3} - 1494x_{n+2}]$$

$$11) \frac{1}{49} [86y_{3n+4} - 240x_{3n+4} + 258y_{n+2} - 720x_{n+2}]$$

$$12) \frac{1}{147} [86y_{3n+5} - 1408x_{3n+4} + 258y_{n+3} - 4224x_{n+2}]$$

$$13) \frac{1}{147} [498y_{3n+4} - 240x_{3n+5} + 1494y_{n+2} - 720x_{n+3}]$$

$$14) \frac{1}{49} [498y_{3n+5} - 1408x_{3n+5} + 1494y_{n+3} - 4224x_{n+3}]$$

$$15) \frac{1}{392} [1408y_{3n+4} - 240y_{3n+5} + 4224y_{n+2} - 720y_{n+3}]$$

D. Each of the following expressions is a biquadratic integer.

$$16) \frac{1}{49} [18y_{4n+4} - 32x_{4n+4} + 72y_{2n+2} - 128x_{2n+2} + 294]$$

$$17) \frac{1}{49} [18x_{4n+5} - 86x_{4n+4} + 72x_{2n+3} - 344x_{2n+2} + 294]$$

$$18) \frac{1}{147} [9x_{4n+6} - 249x_{4n+4} + 36x_{2n+4} - 996x_{2n+2} + 882]$$

$$19) \frac{1}{147} [18y_{4n+5} - 240x_{4n+4} + 72y_{2n+3} - 960x_{2n+2} + 882]$$

$$20) \frac{1}{833} [18y_{4n+6} - 1408x_{4n+4} + 72y_{2n+4} - 5632x_{2n+2} + 4998]$$

$$21) \frac{1}{147} [86y_{4n+4} - 32x_{4n+5} + 344y_{2n+2} - 128x_{2n+3} + 882]$$

$$22) \frac{1}{833} [498y_{4n+4} - 32x_{4n+6} + 1992y_{2n+2} - 128x_{2n+4} + 4998]$$

$$23) \frac{1}{392} [240y_{4n+4} - 32y_{4n+5} + 960y_{2n+2} - 128y_{2n+3} + 2352]$$

$$24) \frac{1}{2352} [1408y_{4n+4} - 32y_{4n+6} + 5632y_{2n+2} - 128y_{2n+4} + 14112]$$

$$25) \frac{1}{49} [86x_{4n+6} - 498x_{4n+5} + 344x_{2n+4} - 1992x_{2n+3} + 294]$$

$$26) \frac{1}{49} [86y_{4n+5} - 240x_{4n+5} + 344y_{2n+3} - 960x_{2n+3} + 294]$$

$$27) \frac{1}{147} [86y_{4n+6} - 1408x_{4n+5} + 344y_{2n+4} - 5632x_{2n+3} + 882]$$

$$28) \frac{1}{147} [498y_{4n+5} - 240x_{4n+6} + 1992y_{2n+3} - 960x_{2n+4} + 882]$$

$$29) \frac{1}{49} [498y_{4n+6} - 1408x_{4n+6} + 1992y_{2n+4} - 5632x_{2n+4} + 294]$$

$$30) \frac{1}{392} [1408y_{4n+5} - 240y_{4n+6} + 5632y_{2n+3} - 960y_{2n+4} + 2352]$$

E. Each of the following expression is a quintic integer:

$$1) \frac{1}{49} [18y_{5n+5} - 32x_{5n+5} + 90y_{3n+3} - 160x_{3n+3} + 180y_{n+1} - 320x_{n+1}]$$

$$2) \frac{1}{49} [18x_{5n+6} - 86x_{5n+5} + 90x_{3n+4} - 430x_{3n+3} + 180x_{n+2} - 860x_{n+1}]$$

$$3) \frac{1}{147} [9x_{5n+7} - 249x_{5n+5} - 45x_{3n+5} - 1245x_{3n+3} + 90x_{n+3} - 2490x_{n+1}]$$

- 4)  $\frac{1}{147} [18y_{5n+6} - 240x_{5n+5} + 90y_{3n+4} - 1200x_{3n+3} + 180y_{n+2} - 2400x_{n+1}]$
- 5)  $\frac{1}{833} [18y_{5n+7} - 1408x_{5n+5} + 90y_{3n+5} - 7040x_{3n+3} + 180y_{n+3}]$
- 6)  $\frac{1}{147} [86y_{5n+5} - 32x_{5n+6} + 430y_{3n+3} - 160x_{3n+4} + 860y_{n+1} - 320x_{n+2}]$
- 7)  $\frac{1}{833} [498y_{5n+5} - 32x_{5n+7} + 2490y_{3n+3} - 160x_{3n+5} + 4980y_{n+1} - 320x_{n+3}]$
- 8)  $\frac{1}{392} [240y_{5n+5} - 32y_{5n+6} + 1200y_{3n+3} - 160y_{3n+4} + 2400y_{n+1} - 320y_{n+2}]$
- 9)  $\frac{1}{2352} [1408y_{5n+5} - 32y_{5n+7} + 7040y_{3n+3} - 160y_{3n+5} + 14080y_{n+1} - 320y_{n+3}]$
- 10)  $\frac{1}{49} [86x_{5n+7} - 498x_{5n+6} + 430x_{3n+5} - 2490x_{3n+4} + 860x_{n+3} - 4980x_{n+2}]$
- 11)  $\frac{1}{49} [86y_{5n+6} - 240x_{5n+6} + 430y_{3n+4} - 1200x_{3n+4} + 860y_{n+2} - 2400x_{n+2}]$
- 12)  $\frac{1}{147} [86y_{5n+7} - 1408x_{5n+6} + 430y_{3n+5} - 7040x_{3n+4} + 860y_{n+3} - 14080x_{n+2}]$
- 13)  $\frac{1}{147} [498y_{5n+6} - 240x_{5n+7} + 2490y_{3n+4} - 1200x_{3n+5} + 4980y_{n+2} - 2400x_{n+3}]$
- 14)  $\frac{1}{49} [498y_{5n+7} - 1408x_{5n+7} + 2490y_{3n+5} - 7040x_{3n+5} + 4980y_{n+3} - 14080x_{n+3}]$
- 15)  $\frac{1}{392} [1408y_{5n+6} - 240y_{5n+7} + 7040y_{3n+4} - 1200y_{3n+5} + 14080y_{n+2} - 24840y_{n+3}]$

F. Relations among the solutions are given below:

- 1)  $x_{n+2} = y_{n+1} + 3x_{n+1}$
- 2)  $x_{n+3} = 6y_{n+1} + 17x_{n+1}$
- 3)  $y_{n+2} = 3y_{n+1} + 8x_{n+1}$
- 4)  $y_{n+3} = 17y_{n+1} + 48x_{n+1}$
- 5)  $x_{n+3} = 6x_{n+2} - x_{n+1}$
- 6)  $y_{n+2} = 3x_{n+2} - x_{n+1}$
- 7)  $y_{n+3} = 17x_{n+2} - 3x_{n+1}$
- 8)  $2y_{n+2} = x_{n+3} - x_{n+1}$
- 9)  $6y_{n+3} = 17x_{n+3} - x_{n+1}$
- 10)  $3y_{n+3} = 17y_{n+2} + 8x_{n+1}$
- 11)  $3x_{n+3} = y_{n+1} + 17x_{n+2}$
- 12)  $3y_{n+2} = y_{n+1} + 8x_{n+2}$

- 13)  $y_{n+3} = y_{n+1} + 16x_{n+2}$
- 14)  $17y_{n+2} = 3y_{n+1} + 8x_{n+3}$
- 15)  $17y_{n+3} = y_{n+1} + 48x_{n+3}$
- 16)  $y_{n+3} = 6y_{n+2} - y_{n+1}$
- 17)  $y_{n+2} = x_{n+3} - 3x_{n+2}$
- 18)  $y_{n+3} = 3x_{n+3} - x_{n+2}$
- 19)  $y_{n+3} = 3y_{n+2} + 8x_{n+2}$
- 20)  $3y_{n+3} = y_{n+2} + 8x_{n+3}$

**III.REMARKABLE OBSERVATION**

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 8X^2 = 9604$	$(18x_{n+1} - 4y_{n+1}, 18y_{n+1} - 32x_{n+1})$
2	$Y^2 - 8X^2 = 9604$	$(30x_{n+1} - 4x_{n+2}, 18x_{n+2} - 86x_{n+1})$
3	$Y^2 - 8X^2 = 86436$	$(88x_{n+1} - 2x_{n+3}, 9x_{n+3} - 249x_{n+1})$
4	$Y^2 - 8X^2 = 86436$	$(86x_{n+1} - 4y_{n+2}, 18y_{n+2} - 240x_{n+1})$
5	$Y^2 - 8X^2 = 2775556$	$(498x_{n+1} - 4y_{n+3}, 18y_{n+3} - 1408x_{n+1})$
6	$Y^2 - 8X^2 = 86436$	$(18x_{n+2} - 30y_{n+1}, 86y_{n+1} - 32x_{n+2})$
7	$Y^2 - 8X^2 = 2775556$	$(18x_{n+3} - 176y_{n+1}, 498y_{n+1} - 32x_{n+3})$
8	$Y^2 - 8X^2 = 614656$	$(18y_{n+2} - 86y_{n+1}, 240y_{n+1} - 32y_{n+2})$
9	$Y^2 - 8X^2 = 22127616$	$(18y_{n+3} - 498y_{n+1}, 1408y_{n+1} - 32y_{n+3})$
10	$Y^2 - 8X^2 = 9604$	$(176x_{n+2} - 30x_{n+3}, 86x_{n+3} - 498x_{n+2})$
11	$Y^2 - 8X^2 = 9604$	$(86x_{n+2} - 30y_{n+2}, 86y_{n+2} - 240x_{n+2})$
12	$Y^2 - 8X^2 = 86436$	$(498x_{n+2} - 30y_{n+3}, 86y_{n+3} - 1408x_{n+2})$
13	$Y^2 - 8X^2 = 86436$	$(86x_{n+3} - 176y_{n+2}, 498y_{n+2} - 240x_{n+3})$
14	$Y^2 - 8X^2 = 9604$	$(498x_{n+3} - 176y_{n+3}, 498y_{n+3} - 1408x_{n+3})$
15	$Y^2 - 8X^2 = 614656$	$(86y_{n+3} - 498y_{n+2}, 1408y_{n+2} - 240y_{n+3})$

B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$49Y - 8X^2 = 9604$	$(18x_{n+1} - 4y_{n+1}, 18y_{2n+2} - 32x_{2n+2} + 98)$
2	$49Y - 8X^2 = 9604$	$(30x_{n+1} - 4x_{n+2}, 18x_{2n+3} - 86x_{2n+2} + 98)$
3	$147Y - 8X^2 = 86436$	$(88x_{n+1} - 2x_{n+3}, 9x_{2n+4} - 249x_{2n+2} + 294)$
4	$147Y - 8X^2 = 86436$	$(86x_{n+1} - 4y_{n+2}, 18y_{2n+3} - 240x_{2n+2} + 294)$
5	$833Y - 8X^2 = 2775556$	$(498x_{n+1} - 4y_{n+3}, 18y_{2n+4} - 1408x_{2n+2} + 1666)$
6	$147Y - 8X^2 = 86436$	$(18x_{n+2} - 30y_{n+1}, 86y_{2n+2} - 32x_{2n+3} + 294)$
7	$833Y - 8X^2 = 2775556$	$(18x_{n+3} - 176y_{n+1}, 498y_{2n+2} - 32x_{2n+4} + 1666)$
8	$392Y - 8X^2 = 614656$	$(18y_{n+2} - 86y_{n+1}, 240y_{2n+2} - 32y_{2n+3} + 784)$
9	$2352Y - 8X^2 = 22127616$	$(18y_{n+3} - 498y_{n+1}, 1408y_{2n+2} - 32y_{2n+4} + 4704)$
10	$49Y - 8X^2 = 9604$	$(176x_{n+2} - 30x_{n+3}, 86x_{2n+4} - 498x_{2n+3} + 98)$
11	$49Y - 8X^2 = 9604$	$(86x_{n+2} - 30y_{n+2}, 86y_{2n+3} - 240x_{2n+3} + 98)$
12	$147Y - 8X^2 = 86436$	$(498x_{n+2} - 30y_{n+3}, 86y_{2n+4} - 1408x_{2n+3} + 294)$
13	$147Y - 8X^2 = 86436$	$(86x_{n+3} - 176y_{n+2}, 498y_{2n+3} - 240x_{2n+4} + 294)$
14	$49Y - 8X^2 = 9604$	$(498x_{n+3} - 176y_{n+3}, 498y_{2n+4} - 1408x_{2n+4} + 98)$
15	$392Y - 8X^2 = 614656$	$(86y_{n+3} - 498y_{n+2}, 1408y_{2n+3} - 240y_{2n+4} + 784)$

1) Special Cases

a)  $P_y^{10}(t_{3,x+1})^2 = 153P_x^6(t_{3,y})^2 + 8(t_{3,y})^2(t_{3,x+1})^2$

b)  $9P_y^6(t_{3,x})^2 = 17P_x^{10}(t_{3,y+1})^2 + 8(t_{3,x})^2(t_{3,y+1})^2$

c)  $P_y^{10}(t_{3,2x-2})^2 = 17(6P_{x-1}^4)^2(t_{3,y})^2 + 8(t_{3,y})^2(t_{3,2x-2})^2$

d)  $36P_{y-1}^8(t_{3,x})^2 = 17P_x^{10}(t_{3,2y-2})^2 + 8(t_{3,x})^2(t_{3,2y-2})^2$

e)  $9P_y^6(t_{3,2x-2})^2 = 17(36P_{x-1}^8)(t_{3,y+1})^2 + 8(t_{3,2x-2})^2(t_{3,y+1})^2$

f)  $(6P_{y-1}^4)^2(t_{3,x+1})^2 = 17(3P_x^3)^2(t_{3,2y-2})^2 + 8(t_{3,x+1})^2(t_{3,2y-2})^2$



#### IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations  $y^2 = 8x^2 + 49$ . As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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