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Three Connected Domination in a Graph

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Abstract: Claude Berge [1] introduced the concept of strong stable set S in a graph. These sets are independent and any vertex outside S can have at most one neighbour in S. This concept was generalized by E. Sampathkumar and L. Pushpalatha [5]. A maximal independent set is a minimal dominating set. What type of domination will result from maximal semi-strong sets? This new type of domination which we call us -Three-connected domination is initiated and studied in this paper.
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I. INTRODUCTION

Let $G = (V, E)$ be a simple, finite, undirected graph. A subset S of $V(G)$ is called a strong stable set of G if $| N[v] \cap S | \leq 1$ for v in $V(G)$. It can be easily seen that such a sets is independent and the distance between any two vertices of S greater than equal to three. That is, the strong stable sets is a 2-packing. Generalising this concept, E. Sampathkumar and L. Pushpa Latha [5] introduced the concept of semi-strong sets. A subset S of $V(G)$ is called semi-strong stable if $| N(v) \cap S | \leq 1$ for every v in $V(G)$. A strong stable set is semi-strong stable but the converse is not true. For example, in C_5, any two consecutive vertices is a semi- strong stable set. If S is a semi-strong stable set, then any component of S is either K_1 or K_2 and the distance between any two points of S is not equal to two. A maximal semi-strong stable set gives rise to a new type of domination and this is studied in this paper.

II. THREE-CONNECTED DOMINATING SET

1) *Definition 2.1:* Let S be a subset of $V(G)$. For any $u \in V - S$, if there exists $v \in V(G)$, $v \neq u$ such that v is adjacent with u and v is adjacent with a vertex of S, (that is, for any $u \in V(G)$ and $w \in S$ such that uvw is a path P_3), then S is called a 3-connected dominating set of G.
2) *Remark 2.2:* Any 3-connected dominating set S of G which is semi-strong is a maximal semi- strong set of G.
3) *Theorem 2.3:* Let S be a subset of $V(G)$ such that for any $u \in V - S$, there exists v and a vertex w in S such that uvw is a path. This property is super hereditary.

Proof

Let S be a subset of $V(G)$ satisfying the hypothesis. Let T be a proper super set of S. Let $u \in V - T$. Then $u \in V - S$. By hypothesis, there exists a vertex v and a vertex w in S such that uvw is a path.

a) *Case 1:* $v \in V - T$. In this case, $u, v \in V - T$ and $w \in T$ (since $w \in S \subset T$). Moreover uvw is a path.
b) *Case 2:* $v \in T - S$ and $u \in V - T$. There exist w in S such that uvw is a path. That is, $u \in V - T$, $v \in T$, $w \in T$ and uvw is a path.
c) *Case 3:* $v \in S$ and $u \in V - T$. There exist $w \in S$ such that uvw is a path. That is, $v \in T$ and $w \in T$ and uvw is a path. In all the three cases, for any $u \in V - T$, there exist $v \in V(G)$, $v \neq u$ and $w \in T$ such that uvw is a path. Therefore the property for maximality of a semi-strong set S is super hereditary.

4) *Remark 2.4:* The above property is called a 3-connected dominating property.
5) *Theorem 2.5:* Any minimal 3-connected dominating set is a maximal semi-strong set.

Proof

Let S be a minimal 3-connected dominating set of G.

a) *Case 1:* Let $u \in V - S$

i) *Subcase 1:* There exists $v \in V - S$ and $w \in S$ such that uvw is a path. Suppose u has at least two neighbours in S. Let $x, y \in S$ such that u is adjacent with x and y.

1. Consider $S - \{x\}$. For any u_1 in $V - (S - \{x\})$, $u_1 \neq x$, $u_1 \in V - S$. There exists v in $V(G)$, $v \neq u_1$ and w in S such that uvw is a path if $w = x$. Then u_1vw is a triangle and not a path, contradiction. Therefore $w \neq x$. Therefore $w \in S - \{x\}$. Therefore there exists $w \in (S - \{x\})$ such that u_1vw is a path.
2. Suppose $u_1 = x$. Then $u \in V - S$ such that u is adjacent with x and adjacent with $y \in (S - \{x\})$. That is, u_1 is adjacent with u and u is adjacent with $y \in (S - \{x\})$. Therefore $S - \{x\}$ is a 3-connected dominating set of G, a contradiction (since S is minimal).

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ii) *Subcase 2:* There exist $v, w \in S$ such that uvw is a path.

1. Suppose u has at least two neighbours say v, x in S. Let $u_1 \in V - (S - \{x\})$.
2. Suppose $u_1 \neq x$. Therefore $u_1 \in V - S$. Hence there exists v in $V(G)$ and w in S such that u_1vw is a path. If $w = x$, then u_1vx is a triangle, a contradiction. Therefore $w = x$. Therefore $w \in S - \{x\}$ and uvw is a path.
3. Suppose $u_1 = x$. In this case u_1 is adjacent with $u \in V - S$ and u is adjacent with $v \in (S - \{x\})$. Also u_1uv is a path. Therefore $S - \{x\}$ is a 3-connected dominating set, a contradiction since S is minimal. Therefore $| N(u) \cap S | \neq 1$.

b) *Case 2:* $u \in S$, Suppose u has at least two neighbours say x, y in S. Consider $S - \{x\}$. Then $x \in V - (S - \{x\})$. x is adjacent with $u \in V(G)$ and u is adjacent with $y \in S - \{x\}$. Therefore xuy is a path. Therefore $S - \{x\}$ is a 3-connected dominating set of $V(G)$, a contradiction. Therefore for any u in S, $| N(u) \cap S | \leq 1$. Hence S is a semi-strong set of G. Since S is a 3-connected dominating set of G and since S is semi-strong set of G, we get that S is a maximal semi- strong set of G.

6) *Theorem 2.6:* Any maximal semi-strong set of G is a minimal 3-connected dominating a set of G.

Proof

Suppose S is a maximal semi-strong set of G. Then S is a 3-connected dominating set of G. Suppose S is not a minimal 3-connected dominating set of G. Therefore there exists a proper subset T of S such that T is a 3-connected dominating set of G. Since S is semi-strong, T is semi-strong. Therefore T is a maximal semi-strong set of G which satisfies 3-connected property. Therefore T is a maximal semi-strong set of G, a contradiction, since S is a proper superset of T and S is a semi-strong set of G. Therefore S is a minimal 3-connected dominating set of G.

7) *Definition 2.7:* The minimum (maximum) cardinality of a minimal 3-connected dominating set of G is called 3-connected domination number of G (upper 3-connected domination number of G) and is denoted by $\boldsymbol{\gamma_{3-c}}$ (G) ($\boldsymbol{\Gamma_{3-c}}$(G)).

8) *Remark 2.8:* Let S be a minimum cardinality of a maximal semi-strong set of G. Then S is a minimal 3-connected dominating set of G. Therefore $\boldsymbol{\gamma_{3-c}}(G) \leq | S | = lss(G) \leq ss(G)$.

9) *Remark 2.9:* Let S be a maximum semi-strong set of G. Therefore S is a minimal 3-connected dominating set of G. Therefore $ss(G) = | S | \leq \boldsymbol{\Gamma_{3-c}}(G)$. Therefore $\boldsymbol{\gamma_{3-c}}(G) \leq lss(G) \leq ss(G) \leq \boldsymbol{\Gamma_{3-c}}(G)$.

10) *Illustration 2.10:* Let G be the graph given in Figure 1:

In this graph, $S_1 = \{u_1, u_2, u_5, u_7, u_8, u_{11}\}$ is a ss-set of G. Hence $ss(G) = 6$. $S_2 = \{u_3, u_6, u_7, u_{11}\}$ is a maximal semi-strong set of G of minimum cardinality. Therefore $lss(G) = 4$. $S_3 = \{u_3, u_6, u_9\}$ is a minimum 3-connected dominating set of G.

Hence $\gamma_{3-c}((G) = 3 \leq lss(G) = 4$. That is, γ_{3-c} $(G) < lss(G)$.

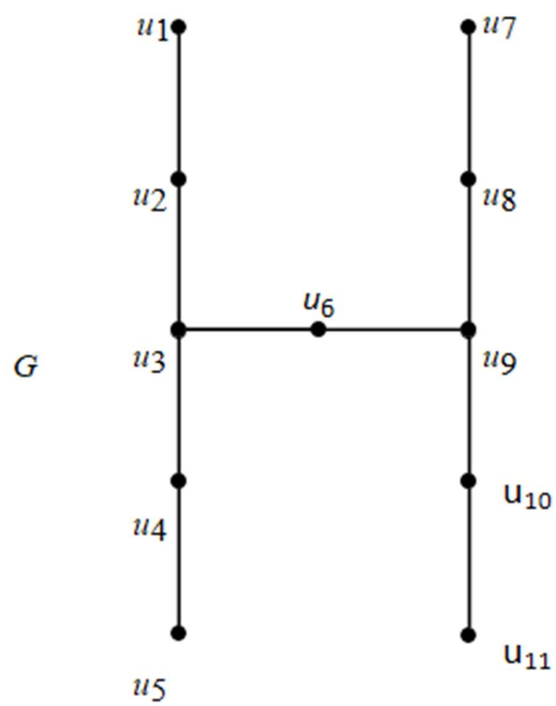

Figure 1: An example graph G for $\boldsymbol{\gamma_{3-c}}(G) < lss(G)$

11) *Theorem 2.11:* Let S be a 3-connected dominating set of G. S is minimal if and only if for any w in S there exists a vertex u in $V - S$ such that any 3-connected path from u to S ends in w.

Proof

Let S be a minimal 3-connected dominating set of G. Let $w \in S$. Then $S - \{w\}$ is not a 3-connected dominating set of G. Therefore there exists u in $V - (S - \{w\})$ such that there is no 3-connected path uv_1w_1 where $v_1 \in V(G)$ and $w_1 \in S - \{x\}$. Since S is a 3-connected dominating set of G, there exists $v_1 \in V(G)$ and w_1 in S such that uv_1w_1 is path. If $w_1 \neq w$, then there exists a 3-connecteed path uv_1w_1 from u to $S - \{w\}$, a contradiction. Therefore $w_1 = w$. Therefore any 3-connected path from u to S is of the form uvw. That is, there exists u in $V - S$ such that any 3-connected path from u to S ends in w.

Conversely, let S be a 3-connected dominating set of G such that for any w in S, there exists u in $V - S$ such that 3-connected path from u to S ends in w.

1) *Claim:* $S - \{w\}$ is not a 3-connected dominating set for any w in S.

Since S is a 3-connected dominating set of G satisfying the above property, there exists u in $V - S$ such that any 3-connected path from u to S must end in w. Therefore $u \in V - (S - \{w\})$, $u \neq v$. Suppose there exists a 3-connected path from u to $S - \{w\}$ say uvw_1, where $w_1 \in S - \{w\}$. Then $w_1 \in S$ and uvw_1 is a path ending in w_1 in S, $w_1 \neq w$, a contradiction. Therefore $S - \{w\}$ is not a 3-connected dominating set of G. Hence the claim.

Therefore S is a minimal 3-connected dominating set of G.

III. THREE-CONNECTED PATH IRREDUNDANCE

1) *Definition 3.1:* Let S be a subset of $V(G)$ such that for any w in S, there exists a u in $V - S$ such that any 3-connected path from u to S ends in w. Then S is called a 3-connected path irredundant set of G.
2) *Theorem 3.2:* The above property of a set S is hereditary.

Proof

Let S be a subset of $V(G)$ satisfying the above property. Let T be a proper subset of S.

Let $w \in T$. Then $w \in S$. Therefore there exist $u \in V - S$ such that any 3-connected path from u to S ends in w. Therefore $u \in V - T$. Suppose there exists a 3-connected path such that $w_1 \in T$,

$w \neq w_1$. Then $w_1 \in S$. Therefore there exists a 3-connected path from u to w_1 in S, a contradiction. Therefore $w_1 = w$. Hence T is a subset of $V(G)$ satisfying the above property. Hence the theorem.

3) *Definition 3.3:* Let S be a 3-connected path set of G. The minimum (maximum) cardinality of a maximal 3-connected path irredundant set of G is called 3-connected path irredundant number of G (upper 3-connected path irredundant number of G) is denoted by $ir_{3\text{-}C}(G)$ $(IR_{3\text{-}C}(G))$.
4) *Remark 3.4:* Any 3-consecutive dominating set of G is minimal if and only if it a 3-consecutive path irredundant set of G.
5) *Theorem 3.5:* Every minimal 3-connected dominating set of G is a maximal 3-connected path irredundant set of G.

Proof

Let S be a minimal 3-connected dominating set of G. Then S satisfies the property that for every w in S, there exists u in $V - S$ such that any 3-connected path from u to S ends in w. Therefore S is a 3-connected path irredundant set of G. Suppose S is not a maximal 3-connected path irredundant set of G.

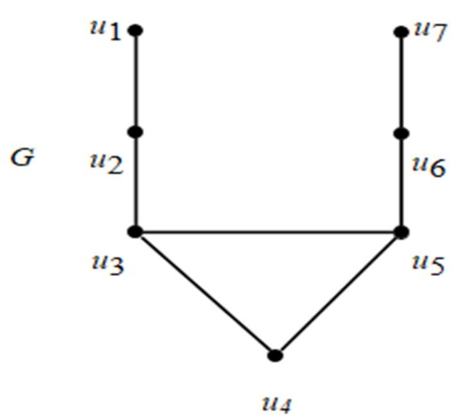

Figure 2: An example graph G for which $ir_{3\text{-}C}(G) < \gamma_{3-C}(G)$

Since 3-connected path irredundant is hereditary, it is enough to consider 1-maximality. Since S is not maximal, there exists u in $(V - S)$ such that $S \cup \{u\}$ is 3-connected path irredundant set of G. Therefore for any x in $S \cup \{u\}$, there exist y in $V - (S \cup \{u\})$ such that any 3-connected path from y in $S \cup \{u\}$ ends in x. Take $x = u$. Then there exists y in $V - (S \cup \{u\})$ such that any 3-connected path from y in $S \cup \{u\}$ ends in u. That is, there exists y in $V - S$ such that any 3-connected path from y to S does not end in any vertex of S, that is, S does not satisfy 3-connected path irredundant condition, a contradiction. Therefore S is a maximal 3-connected path irredundant set of G.

6) *Remark 3.6:* For any graph G, $ir_{3\text{-}C}(G) \leq \boldsymbol{\gamma_{3-C}}(G) \leq lss(G) \leq ss(G) \leq \boldsymbol{\Gamma_{3-C}}(G) \leq IR_{3\text{-}C}(G)$.

7) *Remark 3.7:* In the following example, $ir_{3\text{-}C}(G) < \boldsymbol{\gamma_{3-C}}(G)$. Let G be the graph given in Figure 2. The set $S_1 = \{u_2, u_4, u_6\}$ is a minimum 3-connected dominating set of G. Therefore $\gamma_{3-C}(G) = 3$.

The set $S_2 = \{u_3, u_5\}$ is maximum 3-connected path irredundant set of G. $ir_{3\text{-}C}(G) = 2$.

Therefore $ir_{3\text{-}C}(G) < \gamma_{3-C}(G)$

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