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An Advance Class of Analytic Functions with Fekete-Szegö Inequality using subordination Method

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***Abstract:** In this Paper we have introduced an advance class of analytic functions along with its subclasses by using principle of subordination and as so obtained sharp upper Bound of the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to these classes. Extremal functions are also investigated.*

***Keywords:** Bounded functions, Close to convex function, extremal function, Inverse Starlike functions, Starlike functions, Univalent functions.*

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

In the unit disc $= \{z: |z| < 1|\}$, $\mathcal{S}$ be the class of analytic univalent functions in $\mathbb{E}$.

Bieber Bach [7] proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. and Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.. With the above known estimates this inequality plays an important role to determining estimates of higher coefficients for some sub classes $\mathcal{S}$ {Chhichra [11], Babalola [6]}

Using Löwner's method [5] In 1933, Fekete and szego investigated a well known relation between a_3 and a_2^2 for the class $\mathcal{S}$

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & , if\ \mu \leq 0 \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)} & , if\ 0 \leq \mu \leq 1 \\ 4\mu - 3 & , if\ \mu \geq 1 \end{cases} \qquad (1.2)$$

Let us define some subclasses of $\mathcal{S}$

Let $\mathcal{K}$ denotes subclasses of $\mathcal{S}$ of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.3)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exist $g(z) \in S^*$ such that

$$Re\frac{((zf'(z))}{g(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

The class of close to convex functions introduced by Kaplan [17],and he proved that close to convex functions are univalent.

$$S^*(A,B) = \{ f(z) \in \mathcal{A} ; \frac{((zf'(z))}{g(z)} \prec \frac{1+Az}{1+Bz}, \ -1 \leq B \leq A \leq 1, z \in \mathbb{E} \} \qquad (1.5)$$

Where $S^*(A,B)$ is a subclass of S^*.

For strongly alpha quasi-convex functions Fekete-Szegö problem was studied by Abdel-Gawad [3]. The upper bound of $|a_3 - \mu a_2^2|$ for different functions in the class S has been investigated by many authors including Goel and Mehrok [13] and recently by Al-Shaqsi and Darus [4] Hayami and Owa [16], Al-Abbadi and Darus [10].

Gurmeet singh et al. [3] introduced the class of inverse Starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ satisfying the condition

$$Re\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right) > 0, \ z \in E \quad \text{i.e.} \frac{zf(z)}{2\int_0^z f(z)dz} < \frac{1+z}{1-z} \qquad (1.6)$$

Gandhi et al. [14] established a new class of analytic functions with Fekete-szego inequality using subordination method.

Here introduce the class $\mathcal{A}$, of Univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\left[\frac{z\{zf(z)\}'}{2f(z)}\right] \prec \left(\frac{1+z}{1-z}\right)^{\alpha} ; \alpha > 0 \qquad (1.7)$$

And subclass consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\left[\frac{z\{zf(z)\}'}{2f(z)}\right] \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha} ; \ -1 \le B \le A \le 1 ; \alpha > 0 \qquad (1.8)$$

Here, Symbol $\prec$ stands for subordination, defined as follows:

A. Principle of Subordination

If $f(z)$ and $F(z)$ are two functions which are analytic in $\mathbb{E}$, then $f(z)$ is called a subordinate to $F(z)$ in $\mathbb{E}$, if there exists a function $w(z)$ which is analytic in $\mathbb{E}$ satisfying the conditions

(i) $w(0) = 0$ and (ii) $|w(z)| < 1$

such that $f(z) = F(w(z))$, where $z \in \mathbb{E}$ and we denote it as $f(z) \prec F(z)$.

Let $\mathcal{U}$ denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1 \qquad (1.9)$$

With $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$.

II. RESULTS AND DISCUSSION

1) *Theorem 1:* If $f(z) \in \mathcal{A}$, then the result

$$|a_3 - \mu a_2^2| \le \begin{cases} 10\alpha^2 - 16\mu\alpha^2 & , if \ \mu \le \frac{5\alpha-1}{8\alpha} \quad (1.10) \\ 2\alpha & , if \ \frac{5\alpha-1}{8\alpha} \le \mu \le \frac{5\alpha+1}{8\alpha} \quad (1.11) \\ 16\mu\alpha^2 - 10\alpha^2 & , if \ \mu \ge \frac{5\alpha+1}{8\alpha} \quad (1.12) \end{cases}$$

is sharp.

Proof: By using expantion method (1.7) leads to

$$1 + \frac{1}{2} a_2 z + (a_3 - \frac{1}{2} a_2^2) z^2 + \text{------} = 1 + 2\alpha c_1 z + 2\alpha(c_2 + \alpha c_1^2) z^2 + \text{-----} \qquad (1.13)$$

After Identifying the terms we have

$$|a_3 - \mu a_2^2| \le |2\alpha c_2 + 10\alpha^2 c_1^2 - 16\mu\alpha^2 c_1^2|$$

This leads to

$$|a_3 - \mu a_2^2| \le 2\alpha + [\,|10\alpha^2 - 16\mu\alpha^2| - 2\alpha]\,|c_1|^2 \qquad (1.14)$$

Case I: If $\mu \le \frac{5}{8}$, then (1.14) leads to

$$|a_3 - \mu a_2^2| \le 2\alpha + [\,(10\alpha^2 - 2\alpha) - 16\mu\alpha^2 |c_1|^2 \qquad (1.15)$$

Subcase I(a): If $\mu \le \frac{5\alpha-1}{8\alpha}$, then (1.15) leads to

$$|a_3 - \mu a_2^2| \le 10\alpha^2 - 16\mu\alpha^2 \qquad (1.16)$$

Subcase I(b) : If $\mu \ge \frac{5\alpha-1}{8\alpha}$, then (1.15) leads to

$$|a_3 - \mu a_2^2| \le 2\alpha \qquad (1.17)$$

Case II : If $\mu \ge \frac{5}{8}$, then (1.14) leads to

$$|a_3 - \mu a_2^2| \le 2\alpha + [\,16\mu\alpha^2 - (10\alpha^2 + 2\alpha)]\,|c_1|^2 \qquad (1.18)$$

Subcase II(a): If $\mu \le \frac{5\alpha+1}{8\alpha}$, then (1.18) leads to

$$|a_3 - \mu a_2^2| \le 2\alpha \qquad (1.19)$$

Subcase II(b): If $\mu \ge \frac{5\alpha+1}{8\alpha}$, then (1.18) leads to

$$|a_3 - \mu a_2^2| \le 16\mu\alpha^2 - 10\alpha^2 \qquad (1.20)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \le 2\alpha \quad , \quad if \ \frac{5\alpha-1}{8\alpha} \le \mu \le \frac{5\alpha+1}{8\alpha} \qquad (1.21)$$

This completes the theorem, therefore the result is sharp.

Extremal function for the first and third inequality is given by

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$$f_1(z) = \frac{z}{(1-\alpha z)^4} \qquad (1.22a)$$

And Extremal function for the second inequality is given by

$$f_2(z) = \frac{z}{(1-\alpha z^2)^2} \qquad (1.22b)$$

2) *Theorem 2 :* If $\boldsymbol{f(z)} \in \boldsymbol{\mathcal{A}}$, then the result

$|\boldsymbol{a_3} - \mu \boldsymbol{a_2^2}| \leq$

$$\begin{cases} (A-B)(2A-3B)\alpha^2 - 4\mu\alpha^2(A-B)^2 & , if\ \mu \leq \dfrac{(2A-3B)\alpha - 1}{4(A-B)\alpha} \quad (1.23a) \\ (A-B)\alpha & , if\ \dfrac{(2A-3B)\alpha - 1}{4(A-B)\alpha} \leq \mu \leq \dfrac{(2A-3B)\alpha + 1}{4(A-B)\alpha} \quad (1.23b) \\ 4\mu\alpha^2(A-B)^2 - (A-B)(2A-3B)\alpha^2 & , if\ \mu \leq \dfrac{(2A-3B)\alpha + 1}{4(A-B)\alpha} \quad (1.23c) \end{cases}$$

is sharp.

Proof: By using expantion method (1.8) leads to

$$1 + \frac{1}{2}a_2 z + (a_3 - \frac{1}{2}a_2^2) z^2 + \text{------} = 1 + (A-B)\alpha c_1 z + (A-B)\alpha(c_2 - B\alpha c_1^2) z^2 + \text{----} \qquad (1.24)$$

After Identifying the terms in (1.24) we have

$$|a_3 - \mu a_2^2| \leq |(A-B)\,\alpha(c_2 - B\alpha c_1^2) + 2(A-B)^2\alpha^2 c_1^2 - 4\mu(A-B)^2\alpha^2 c_1^2|$$

This leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha + \{|2(A-B)^2\alpha^2 - B(A-B)\alpha^2 - 4\mu(A-B)^2\alpha^2| - (A-B)\alpha\}|c_1|^2 \qquad (1.25)$$

here two cases arise:

Case I: If $\mu \leq \frac{2A-3B}{4(A-B)}$, then (1.25) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha + [(A-B)\{(2A-3B)\alpha - 1\}\alpha - 4\mu(A-B)^2\alpha^2]|c_1|^2 \qquad (1.26)$$

Under this case (1.26) two subcases arise:

Subcase I(a): If $\mu \leq \frac{(2A-3B)\alpha-1}{4(A-B)\alpha}$, then (1.26) leads to

$$|a_3 - \mu a_2^2| \leq \{(A-B)(2A-3B)\alpha - 4\mu(A-B)^2\alpha^2 \qquad (1.27)$$

Subcase I(b): If $\mu \geq \frac{(2A-3B)\alpha-1}{4(A-B)\alpha}$, then (1.26) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha \qquad (1.28)$$

Case II: If $\mu \geq \frac{2A-3B}{4(A-B)}$, then (1.25) leads to

$$|a_3 - \mu a_2^2| \leq (A-B) + [4\mu(A-B)^2\alpha^2 - (A-B)\{(2A-3B)\alpha + 1\}\alpha]|c_1|^2 \qquad (1.29)$$

Under this case again two subcases arise:

Subcase II(a): If $\mu \leq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha}$, then (1.29) leads to

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha \qquad (1.30)$$

Subcase II(B): $\mu \geq \frac{(2A-3B)\alpha+1}{4(A-B)\alpha}$, then (1.29) leads to

$$|a_3 - \mu a_2^2| \leq \{4\mu(A-B)^2\alpha^2 - (A-B)(2A-3B)\alpha^2 \qquad (1.31)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq (A-B)\alpha \quad , if\ \frac{(2A-3B)\alpha - 1}{4(A-B)\alpha} \leq \mu \leq \frac{(2A-3B)\alpha + 1}{4(A-B)\alpha} \qquad (1.32)$$

This completes the theorem therefore, the result is sharp.

Extremal function for the first and third inequality is given by

$$f_1(z) = z\left\{1 + \left(\frac{2A-2B}{2A-3B}\right) z\alpha\right\}^{2A-3B} \qquad (1.33a)$$

And for the for second inequality is given by

$$f_2(z) = \frac{z}{(1-\alpha z^2)^{A-B}} \qquad (1.33b)$$

III. CONCLUDING REMARKS AND COROLLARIES

1) *Corollary 1.1:* Taking α = 1 in the theorem 1, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 10 - 16\mu & , if\ \mu \leq \frac{1}{2} & (1.34) \\ 2 & , if\ \frac{1}{2} \leq \mu \leq \frac{3}{4} & (1.35) \\ 16\mu - 10 & , if\ \mu \geq \frac{3}{4} & (1.36) \end{cases}$$

These estimates were derived by Keogh and Merkes [1] and the results are for the class of univalent convex functions.

Further if we take A = 1 and B = -1 ($-1 \leq B \leq A \leq 1$) in the result of theorem 2 , we get the result of theorem 1, therefore our result for the theorem 2 reduces to the result of the theorem1. Hence theorem 2 is the generalization of theorem 1. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 2, we get the extremal function of theorem 1. The extremal function given by [(1.22a),(1.22b)] increases as α increases and decreases as α decreases and the extremal function given by [(1.33a),(1.33b)] also increases and decreases as α increases and decreases respectively. Hence extremal function is an increasing function.

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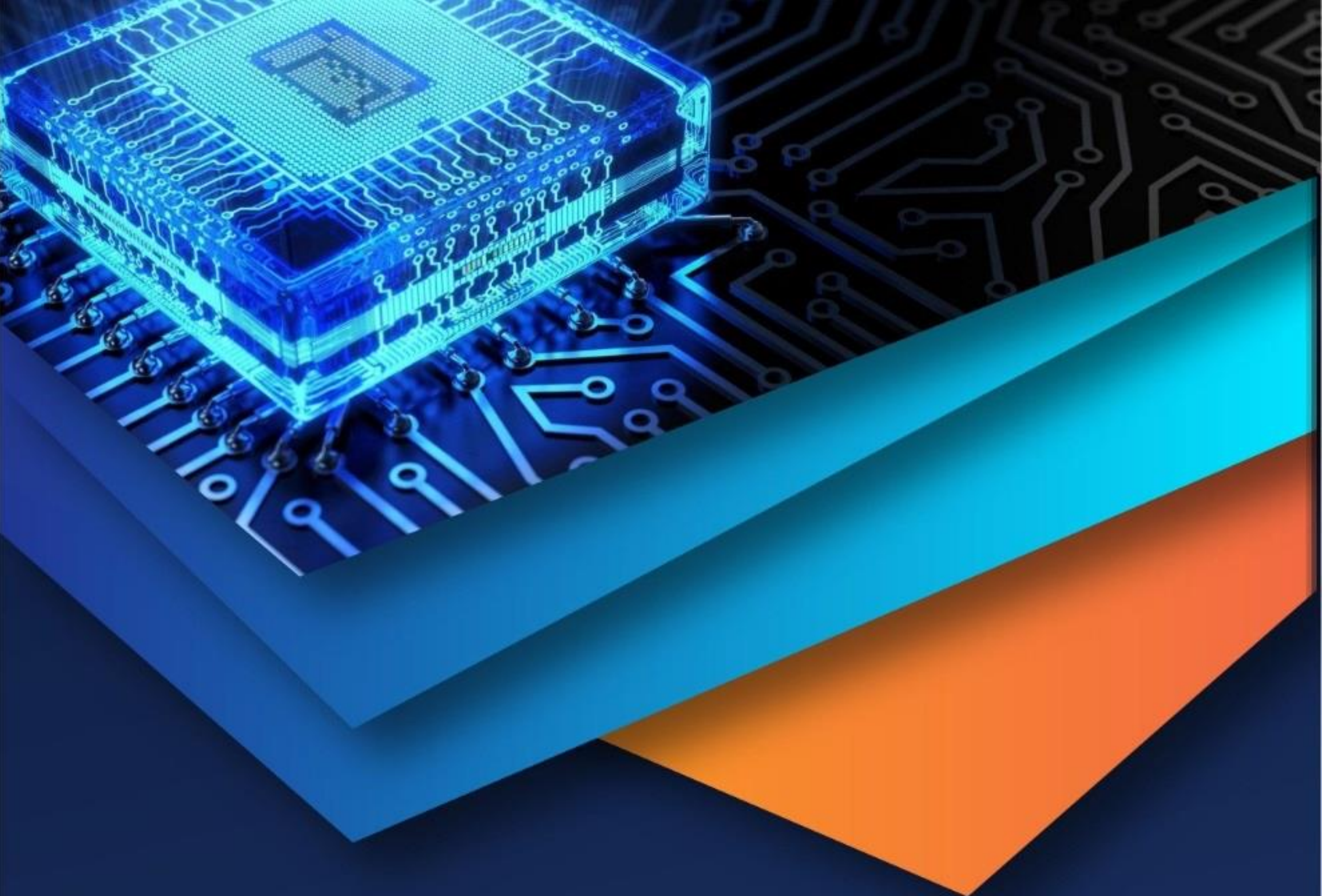

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