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Roman Coloring of Cycle related Graphs

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***Abstract:** Suresh Kumar [7] introduced the Roman coloring, and the Roman Chromatic number motivated from the traditional Roman military defence strategy. In this paper, we investigate the Roman coloring and obtain the Roman Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph.*
***Keyword:** Graph, Roman Coloring, Roman Chromatic Number, Wheel graph, Helm graph, Gear graph, Flower graph, Friendship graph, Crown graph, Web graph*

I. INTRODUCTION

The majority of early graph theory research on graph coloring pays attention only to finding some possible solution to the Four Color Conjecture. After Appel and Haken gave a computer verification proof of the Four Color Conjecture, research focus on graph coloring was shifted to vertex coloring that satisfies some specified property for the induced edge coloring [5]. The coloring is also played an important role in combinatorial optimization and critical graphs were crucial in the Chromatic number Theory [8, 9, 10, 11, 12].

Jason Robert Lewis [1] suggested several new graph parameters in his Doctoral Thesis. Several studies were made in applying such parameters to Roman defense strategy [2, 3, 4, 5, 6]. The basic idea was that in a specified city, if the streets are considered as the edges of a graph and the meeting points of the streets, called the junctions, as the edges of the graph, then we can color each vertex by the number of soldiers deployed at that junction and require that every street (edge) should be guarded by at least one soldier using a strategy that if any street have no soldier, then there must be an adjacent junction with two soldiers so that one among them may be deployed to the former junction in case of emergency. This motivated us to define a new type of graph coloring, Roman Coloring and the related parameter, Roman Chromatic number [7]. S.K.Vaidya [14] studied the total coloring of some cycle related graphs. In this paper, we study the Roman Coloring and obtain the value of Roman number for some special cycle related graphs. For the terms and definitions not explicitly here, refer Harary [13].

We begin by recalling some basic definitions which are useful for the present investigation.

1) *Definition.1.1.*The Wheel graph,$W_n, n \geq 3$, is the join of the graphs C_n and K_1 .That is, W_n is the (n+1)-vertex graph obtained from the graph C_n by adding a new vertex, v and joining it to each of the n vertices of the cycle,C_n.Here we call the vertices corresponding to C_n as rim vertices and the vertex corresponding to K_1(the newly added vertex) is called the apex vertex.
2) *Definition 1.2.* The Helm graph $H_n, n \geq 3$ is the graph obtained from Wheel graph, W_n by adding a pendent edge at each vertex on the rim of the Wheel, W_n.
3) *Definition 1.3.* The closed Helm graph, CH_n, is the graph obtained from a Helm graph $H_n$and adding edges between the pendent vertices.
4) *Definition 1.4.* The Gear graph, G_n, is a graph obtained from Wheel graph, $W_n$by adding an extra vertex between each pair of adjacent vertices on the rim of the Wheel graph W_n.
5) *Definition 1.5.* The Flower graph FL_n is the graph obtained from a Helm graph by joining each pendant vertex to the central vertex of the Helm.
6) *Definition 1.6.* The Friendship graph, F_n can be constructed by joining n copies of the cycle Graph, $C_3$to a common vertex.
7) *Definition 1.7.* The Double Wheel graph,DW_n of size n is composed of $2C_n + K_1$.It consists of two cycles C_n, where vertices of each of these two cycles are connected to a common vertex.
8) *Definition 1.8.* The Crown graph, C_n^+ is obtained from the cycle graph, C_n by adding a pendent edge to each vertex of C_n
9) *Definition1.9.* The Double crown graph, C_n^{++} is the graph obtained from the cycle, C_n, by adding two pendent edge at each vertex of C_n
10) *Definition1.10.* The Web graph is obtained from a Helm by joining the pendent vertices of the Helm to form a cycle and then adding a pendent edge to each vertex of the outer cycle.
11) *Definition1.11.* The floor of a real number x is the largest integer less than or equal to x and it is denoted by$\lfloor x \rfloor$. The ceil of a real number x is the smallest integer greater than or equal to x and it is denoted by $\lceil x \rceil$.

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II. MAIN RESULTS

Let G be a connected graph. Roman coloring of G is an assignment of three colors {0, 1, 2} to the vertices of G such that any vertex with color, 0 must be adjacent to a vertex with color, 2. The color classes will be denoted as V_0, V_1, V_2 which are subsets of V(G) with colors 0,1, 2 respectively.

Weight of a Roman coloring is defined as the sum of all vertex colors. Roman Chromatic number of a graph G is defined as the minimum weight of a Roman coloring on G and is denoted by $R(G)$. A Roman coloring of G with the minimal weight is called a minimal Roman coloring of G.

In this section, we discuss the Roman Coloring of the cycle related Graphs mentioned above. For the terms and definitions not explicitly defined here, reader may refer Harary [13].

1) *Theorem.2.1.* The Wheel graph, W_n, $n \geq 3$ is Roman colourable and $R(W_n) = 2$.

a) *Proof.* Let the central vertex of the Wheel graph, W_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$

Define a coloring function C: V(W_n)→ $\{0, 2\}$ as follows: Assign the color 2 to the central vertex v and assign the color 0 to all the rim vertices. Then this is a Roman coloring of W_n and R(W_n) =$\sum_{v\in V(G)} C(v)$ =2

2) *Theorem. 2.2.* The Helm graph H_n is Roman colourable and $R(H_n) = n + 2$

a) *Proof.* Let the central vertex of the Helm graph $H_n$be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$

Define C : V(H_n)→ $\{0,1,2\}$ as follows:

$C(v) = 2$

C(v_i)= 0 if 1≤ $i \leq n$

C(w_i)= 1 if 1≤ $i \leq n$

Then this coloring is a minimal Roman colouring and $R(H_n) = \sum_{v\in V(G)} C(v) = n + 2$.

3) *Theorem. 2.3.* The Closed Helm graph,CH_n is Roman colourable and $R(CH_n) = n + 2$

a) *Proof:*Let the central vertex of the Helm graph $H_n$be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

Define C : V(CH_n)→ $\{0,1,2\}$ as follows:

$C(v) = 2$

C(v_i)= 0 if 1≤ $i \leq n$

C(w_i)= 1 if 1≤ $i \leq n$

Then this coloring is a minimal Roman colouring and $R(CH_n) = \sum_{v\in V(G)} C(v) = n + 2$

4) *Theorem. 2.4.* The Gear graph,G_n is Roman colourable and $R(G_n) = n + 2$

a) *Proof:* Let the central vertex of the Gear graph,G_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the newly added vertices are $v_1', v_2', v_3',, \ldots\ldots\ldots\ldots v_n'$.

Define C : V(G_n)→ $\{0,1,2\}$ as follows:

$C(v) = 2$

C(v_i)= 0 , 1≤ $i \leq n$

C(v_j')= 1 if 1≤ $j \leq n$

Then this coloring is a minimal Roman colouring and $R(G_n) = \sum_{v\in V(G)} C(v) = n + 2$

5) *Theorem. 2.5.* The Flower graph, FL_n is Roman colourable and R(FL_n) = n+2

a) *Proof:*Let the central vertex of the Helm graph H_n be v and the vertices on the rim are$v_1, v_2, \ldots v_n$ and the pendent vertices corresponding to the cycle are $w_1, w_2, w_3 \ldots\ldots\ldots w_n$.

Define C : V(FL_n)→ $\{0,1,2\}$ as follows:

$C(v) = 2$

C(v_i)= 0 , 1≤ $i \leq n$

C(w_i)= 1 if 1≤ $i \leq n$

Then this coloring is a minimal Roman colouring and R(FL_n) =$\sum_{v\in V(G)} C(v)$ = n+2.

6) *Theorem. 2.6.* The Friendship graph F_n is Roman colourable and R(F_n) =2 .

a) *Proof:* Let the central vertex of the Friendship graph $F_n$be v and let $\{v_{11,}v_{12}\}$ be the vertices of the first copy of C_3, $\{v_{21,}v_{22}\}$ be the vertices of the second copy of C_3,$\{v_{31,}v_{32}\}$ be the vertices of the third copy of C_3 and so on. Let $\{v_{n1,}v_{n2}\}$ be the vertices of the n[th] copy of C_3.

Define C : V(F_n)→ { 0,1,2} as follows.

$C(v) = 2$

C(v_{i1})= 0 if 1≤ $i \leq n$

C(v_{i2})= 0 if 1≤ $i \leq n$

Then this coloring is a minimal Roman colouring and R(F_n) =$\sum_{v\in V(G)} C(v)$ =2.

7) *Theorem.2.7.* The Double Wheel graph,DW_n is Roman colourable and $R(DW_n) = 2$

a) *Proof :* Let v be the apex vertex of the Double Wheel graph, DW_n. Let $\{v_{1,}v_2, v_3 \dots \dots \dots \dots . . v_n\}$ and $\{u_{1,}u_2, u_3 \dots \dots \dots \dots . . u_n\}$ be vertices of inner and outer cycles of C_n.

Let v be the central vertex.

Define C : V(DW_n)→ { 0,1,2} as follows.

$C(v) = 2$

C(v_i)= 0 , 1≤ $i \leq n$

C(u_i)= 0 , 1≤ $i \leq n$

Then this coloring is a minimal Roman colouring and R(DW_n) =$\sum_{v\in V(G)} C(v)$ =2

8) *Theorem. 2.8.* The Crown graph C_n^+ is Roman colourable and R(C_n^+) =$\begin{cases} 2\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) \; if\ n\ is\ even \\ 2\ \left\lfloor\frac{n}{2}\right\rfloor + \left\lceil\frac{n}{2}\right\rceil \; if\ n\ is\ odd. \end{cases}$

a) *Proof:* Let the vertices on the cycle be $v_{1,}v_2$,v_3…………..$v_n$and the pendent vertices corresponding to the cycle be $w_{1,}w_2$,w_3…………w_n.

i) *Case.1.*n ≥ 4 and n is even

Define C : V(C_n^+)→ { 0,1,2} as follows.

C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$

C(v_{2i-1})= 2 if 1≤ $i \leq \frac{n}{2}$

C(w_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$

C(w_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$

This coloring is a minimal Roman colouring and R(C_n^+) =$\sum_{v\in V(G)} C(v)$ =2 $\left[\frac{n}{2}\right]$ + $\left[\frac{n}{2}\right]$.

ii) *Case.2.*n>3 and n is odd

Define C : VC_n^+)→ { 0,1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n-1}{2}$

C(v_{2i-1})= 0 if 1≤ $i \leq \frac{n+1}{2}$

C(w_{2i})= 0 if 1≤ $i \leq \frac{n-1}{2}$

C(w_{2i-1})= 1 if 1≤ $i \leq \frac{n+1}{2}$

Then this coloring is a minimal Roman colouring and R(C_n^+) =$\sum_{v\in V(G)} C(v)$ =2$\left\lfloor\frac{n}{2}\right\rfloor$ +$\left\lceil\frac{n}{2}\right\rceil$.

9) *Theorem. 2.9.* The Double Crown graph, C_n^{++}, is Roman colourable and $R(C_n^{++}) = 2n$

a) *Proof:* Let us label $v_{1,}v_2$,v_3…………..v_n as the vertices of the cycle C_n. Let the pendent edges corresponding to each vertex v_i *be labeled as* $v_{i1,}v_{i2}$

i) *Case.1.*n> 3 and n is even

Define C : V(C_n^{++})→ { 0,1,2} as follows.

C(v_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$

C(v_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$

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C($v_{(2i-1)1}$)= 1 if $1 \leq i \leq \frac{n}{2}$
C($v_{(2i-1)2}$)= 1 if $1 \leq i \leq \frac{n}{2}$
C($v_{(2i)1}$)= 0 if $1 \leq i \leq \frac{n}{2}$
C($v_{(2i)2}$)= 0 if $1 \leq i \leq \frac{n}{2}$
Then this coloring is a minimal Roman colouring and $R(C_n^{++}) = \sum_{v \in V(G)} C(v) = 2n$.

ii) *Case.2.*n> 3 and n is odd
Define C : V(C_n^{++})→ {0,1,2} as follows.
C(v_{2i})= 2 if $1 \leq i \leq \frac{n-1}{2}$
C(v_{2i-1})= 0 if $1 \leq i \leq \frac{n+1}{2}$
C($v_{(2i-1)1}$)= 1 if $1 \leq i \leq \frac{n+1}{2}$
C($v_{(2i-1)2}$)= 1 if $1 \leq i \leq \frac{n+1}{2}$
C($v_{(2i)1}$)= 0 if $1 \leq i \leq \frac{n-1}{2}$
C($v_{(2i)2}$)= 0 if $1 \leq i \leq \frac{n-1}{2}$
Then this coloring is a minimal Roman colouring and $R(C_n^{++}) = \sum_{v \in V(G)} C(v) = 2n$.

10) Theorem. 2.10. The Web graph,Wb_n is Roman colourable and its Roman chromatic number is given by R(Wb_n)

$$= \begin{cases} 2\left(\left\lceil \frac{n+2}{3} \right\rceil + 1\right) + \left(n - \left\lceil \frac{n+2}{3} \right\rceil\right) & if\ n\ is\ even \\ 3\ \left\lceil \frac{n}{2} \right\rceil & if\ n\ is\ odd. \end{cases}$$

a) *Proof:* Let the central vertex of the Web graph, Wb_n be v.
Let the vertices on the innercycle be $v_1, v_2, v_3, \ldots . v_n$ and the vertices on the outercycle be $u_1, u_2, u_3 \ldots \ldots \ldots \ldots . . u_n$ and the pendent vertices be $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

i) *Case.1.n* $= 4$.
Define C : V(Wb_n)→ {0,1,2} as follows:
C(v)=0,
C(v_1)=2 , C(v_2)=0, C(v_3)=0 , C(v_4)=0 .
C(u_1)= 0 , C(u_2)= 0 , C(u_3)= 2 , C(u_4)= 0
C(w_1)=1, C(w_2)=1, C(w_4)=1, C(w_3)=0.
Then this coloring is a minimal Roman colouring and R(Wb_n) =$\sum_{v \in V(G)} C(v)$ =7

ii) *Case.2.n* > 4 and n is even
Define C : V(Wb_n)→ {0,1,2} as follows.
C(v)=2
C(v_i)=0 if $1 \leq i \leq n$
. C(u_{3i-2})= 2 if $1 \leq i \leq \left\lceil \frac{n+2}{3} \right\rceil$
C(u_{3i-1})= 0, if $1 \leq i \leq \left\lceil \frac{n+1}{3} \right\rceil$.
C(u_{3i})= 0 if $1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$
C(w_{3i-2})= 0 if $1 \leq i \leq \left\lceil \frac{n+2}{3} \right\rceil$
C(w_{3i-1})= 1 if $1 \leq i \leq \left\lceil \frac{n+1}{3} \right\rceil$
C(u_{3i})= 1 if $1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$
This coloring is a minimal Roman colouring and
$R(Wb_n) = \sum_{v \in V(G)} C(v) = 2\left(\left\lceil \frac{n+2}{3} \right\rceil + 1\right) + \left(n - \left\lceil \frac{n+2}{3} \right\rceil\right)$

iii) *Case.3.* n>3 and n is odd

Define C : V(Wb_n)→ $\{0,1,2\}$ as follows:

C(v)=2.

C(v_i)= 0 if $1\leq i \leq n$

C(u_{2i})= 2 if $1\leq i \leq \frac{n-1}{2}$

C(u_{2i-1})= 0 if $1\leq i \leq \frac{n+1}{2}$

C(w_{2i})= 0 if $1\leq i \leq \frac{n-1}{2}$

C(w_{2i-1})= 1 if $1\leq i \leq \frac{n+1}{2}$

Then this coloring is a minimal Roman colouring and R(Wb_n) =$\sum_{v\in V(G)} C(v) = 3\left\lceil\frac{n}{2}\right\rceil$.

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