



# IJRASET

International Journal For Research in  
Applied Science and Engineering Technology



---

# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

---

**Volume: 8    Issue: V    Month of publication: May 2020**

**DOI: <http://doi.org/10.22214/ijraset.2020.5341>**

**[www.ijraset.com](http://www.ijraset.com)**

**Call:  08813907089**

**E-mail ID: [ijraset@gmail.com](mailto:ijraset@gmail.com)**

# Proper Roman Coloring of some Cycle related Graphs

Preethi K Pillai<sup>1</sup>, J. Suresh Kumar<sup>2</sup>

<sup>1,2</sup>Post Graduate and Research Department of Mathematics, N.S.S. Hindu College, Changanacherry, Kerala, India 686102

**Abstract:** Suresh Kumar [12] introduced Roman coloring and Roman Chromatic Number of graphs motivated from the traditional Roman military defence strategy. However, it is not a proper coloring. Suresh Kumar and Preethi K Pillai [13] introduced Proper Roman coloring and Proper Roman Chromatic Number of graphs. In this paper, we investigate the Proper Roman colorings and obtain the Proper Roman Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph.

**Keyword:** Graph, Roman coloring, Proper Roman coloring, Proper Roman Chromatic Number, Wheel graph, Helm graph, Gear graph, Flower graph, Friendship graph, Crown graph, Web graph

## I. INTRODUCTION

The majority of early graph theory research on graph coloring pays attention only to find some possible solution to the Four Color Conjecture. After Appel and Haken gave a computer verification proof of the Four Color Conjecture, research focus on graph coloring was shifted to vertex coloring that satisfies some specified property for the induced edge coloring [5]. The coloring is also played an important role in combinatorial optimization problems and critical graphs were crucial in the Chromatic number Theory [7, 8, 9, 10, 11].

Jason Robert Lewis [1] suggested several new graph parameters in his Doctoral Thesis. Several studies were made in applying such parameters to Roman defence strategy [2, 3, 4, 5, 6]. The basic idea was that in a specified city, if the streets are considered as the edges of a graph and the meeting points of the streets, called the junctions, as the edges of the graph, then we can color each vertex by the number of soldiers deployed at that junction and require that every street (edge) should be guarded by at least one soldier using a strategy that if any street have no soldier, then there must be an adjacent junction with two soldiers so that one among them may be deployed to the former junction in case of emergency. Motivated by this, Suresh Kumar [12] defined a new type of graph coloring, namely, Roman Coloring and a related parameter, Roman Chromatic number. However, it is not a proper coloring. Suresh Kumar and Preethi K Pillai [ ] introduced and studied the Roman colorings, which are proper colorings also. In this paper, we investigate the Proper Roman colorings and obtain the Proper Roman Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph. For terms and definitions not explicitly here, refer Harary [13].

We begin by recalling some basic definitions which are useful for the present investigation.

- 1) Definition.1.1. The Wheel graph,  $W_n$ ,  $n \geq 3$ , is the join of the graphs  $C_n$  and  $K_1$ . That is,  $W_n$  is the  $(n+1)$ -vertex graph obtained from the graph  $C_n$  by adding a new vertex,  $v$  and joining it to each of the  $n$  vertices of the cycle,  $C_n$ . Here we call the vertices corresponding to  $C_n$  as rim vertices and the vertex corresponding to  $K_1$  (the newly added vertex) is called the apex vertex.
- 2) Definition. 1.2. The Helm graph  $H_n$ ,  $n \geq 3$  is the graph obtained from Wheel graph,  $W_n$  by adding a pendent edge at each vertex on the rim of the Wheel,  $W_n$ .
- 3) Definition. 1.3. The closed Helm graph,  $CH_n$ , is the graph obtained from a Helm graph  $H_n$  and adding edges between the pendent vertices.
- 4) Definition. 1.4. The Gear graph,  $G_n$ , is a graph obtained from Wheel graph,  $W_n$  by adding an extra vertex between each pair of adjacent vertices on the rim of the Wheel graph  $W_n$ .
- 5) Definition 1.5. The Flower graph  $FL_n$  is the graph obtained from a Helm graph by joining each pendant vertex to the central vertex of the Helm.
- 6) Definition. 1.6. The Friendship graph  $F_n$  can be constructed by joining  $n$  copies of the cycle Graph,  $C_3$  to a common vertex.

- 7) Definition. 1.7. The Double Wheel graph,  $DW_n$  of size  $n$  is composed of  $2C_n + K_1$ . It consists of two cycles  $C_n$ , where vertices of each of these two cycles are connected to a common vertex.
- 8) Definition. 1.8. The Crown graph,  $C_n^+$  is obtained from the cycle graph,  $C_n$  by adding a pendent edge to each vertex of  $C_n$
- 9) Definition. 1.9. The Double crown graph,  $C_n^{++}$  is the graph obtained from the cycle  $C_n$  by adding two pendent edge at each vertex of  $C_n$
- 10) Definition. 1.10. The Web graph is the graph obtained from a Helm graph by joining the pendent vertices of the Helm to form a cycle and then adding a pendent edge to each vertex of the outer cycle.
- 11) Definition. 1.11. The floor of a real number  $x$  is the largest integer less than or equal to  $x$  (that is, the integral part of  $x$ ) and it is denoted by  $\lfloor x \rfloor$ . The ceil of a real number  $x$  is the smallest integer greater than or equal to  $x$  and it is denoted by  $\lceil x \rceil$ .

## II. MAIN RESULTS

Let  $G$  be a connected graph. Proper Roman coloring of a graph  $G$  is an assignment of three colors  $\{0, 1, 2\}$  to the vertices of  $G$  such that adjacent vertices must have distinct colors and any vertex with the color, 0 must be adjacent to a vertex with color, 2. The color classes will be denoted as  $V_0, V_1, V_2$  which are the subsets of  $V(G)$  with colors 0,1,2 respectively.

Weight of a Roman coloring is defined as the sum of all the vertex colors. Proper Roman Chromatic number of a graph  $G$  is defined as the minimum weight of a Roman coloring of  $G$  and is denoted by  $\chi_R(G)$ . A Proper Roman coloring of  $G$  with the minimal weight is called a minimal Proper Roman coloring of  $G$ .

In this section, we discuss the Proper Roman Coloring and obtain the Proper Roman Chromatic number of the cycle related Graphs mentioned above. For the terms and definitions not explicitly defined here, reader may refer Harary [13].

1) *Theorem.2.1.* The Wheel graph,  $W_n$ ,  $n \geq 3$  is not Proper Roman colourable if  $n$  is odd. If  $n$  is even, the Wheel graph,  $W_n$ ,  $n \geq 3$  is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(W_{2n}) = 2 + n$ .

Proof. Since the chromatic number of  $W_n$  is 4 when  $n$  is odd, and only 3 colors, viz  $\{0,1,2\}$ , are available in a Proper Roman Coloring, it follows that the Wheel graph,  $W_n$ ,  $n \geq 3$  is not Proper Roman colourable if  $n$  is odd.

Assume that  $n$  is even. Let the apex vertex of the Wheel graph,  $W_n$  be  $v$  and the vertices on the rim are  $v_1, v_2, \dots, v_n$ .

Define a coloring function  $C : V(W_n) \rightarrow \{0,1,2\}$  as follows:

$$C(v) = 2,$$

$$C(v_{2i}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by  $\chi_R(W_n) = \sum_{v \in V(G)} C(v) = 2 + \frac{n}{2}$

2) *Theorem. 2.2.* Helm graph,  $H_n$ ,  $n \geq 4$  and  $n$  is even is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(H_n) = n+2$ .

Proof: Let the central vertex of the Helm graph  $H_n$  be  $v$  and the vertices on the rim are  $v_1, v_2, \dots, v_n$  and the pendent vertices are  $w_1, w_2, w_3, \dots, w_n$ .

Define  $C : V(H_n) \rightarrow \{0,1,2\}$  as follows:

$$C(v) = 2$$

$$C(v_{2i}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

$$C(w_{2i}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i-1}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by  $\chi_R(H_n) = \sum_{v \in V(G)} C(v) = n+2$ .

3) *Theorem. 2.3.* The Closed Helm graph,  $CH_n$ ,  $n \geq 4$  and  $n$  is even is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(CH_n) = n+2$ .

Proof: Let the central vertex of the Helm graph  $H_n$  be  $v$  and the vertices on the rim are  $v_1, v_2, \dots, v_n$  and the pendent vertices are  $w_1, w_2, w_3, \dots, w_n$ .

Define  $C : V(CH_n) \rightarrow \{0,1,2\}$  as follows:

$$C(v) = 2$$

$$C(v_{2i})= 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i-1})= 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

Then this coloring is a minimal Roman colouring and its chromatic number is given by  $\chi_R(CH_n) = \sum_{v \in V(G)} C(v) = n+2$ .

4) *Theorem. 2.4.* The Gear graph,  $G_n$  is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(G_n) = n + 2$

Proof: Let the central vertex of the Gear graph,  $G_n$  be  $v$  and the vertices on the rim are  $v_1, v_2, \dots, v_n$  and the newly added vertices are  $v_1', v_2', v_3', \dots, v_n'$ .

Define  $C : V(G_n) \rightarrow \{0, 1, 2\}$  as follows:

$$C(v) = 2$$

$$C(v_i)=1, \quad 1 \leq i \leq n$$

$$C(v_j')= 0 \text{ if } 1 \leq j \leq n$$

This coloring is a minimal Proper Roman colouring and its chromatic number is given by  $\chi_R(G_n) = \sum_{v \in V(G)} C(v) = n + 2$

5) *Theorem. 2.5.* Flower graph,  $FL_n, n \geq 4$  and  $n$  is even is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(FL_n) = n + 2$

Proof: Let the central vertex of the Helm graph  $H_n$  be  $v$  and the vertices on the rim are  $v_1, v_2, \dots, v_n$  and the pendent vertices corresponding to the cycle are  $w_1, w_2, w_3, \dots, w_n$ .

Define  $C : V(FL_n) \rightarrow \{0, 1, 2\}$  as follows:

$$C(v) = 2$$

$$C(v_{2i})= 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i-1})= 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by  $\chi_R(FL_n) = \sum_{v \in V(G)} C(v) = n+2$ .

6) *Theorem. 2.6.* The Friendship graph  $F_n$  is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(F_n) = n+2$ .

Proof: Let the central vertex of the Friendship graph  $F_n$  be  $v$  and let  $v_{11}, v_{12}$  be the vertices of the first copy of  $C_3, v_{21}, v_{22}$  be the vertices of the second copy of  $C_3, v_{31}, v_{32}$  be the vertices of the third copy of  $C_3$  and so on. Let  $v_{n1}, v_{n2}$  be the vertices of the  $n^{\text{th}}$  copy of  $C_3$ .

Define  $C : V(F_n) \rightarrow \{0, 1, 2\}$  as follows.

$$C(v) = 2$$

$$C(v_{i1})= 0 \text{ if } 1 \leq i \leq n$$

$$C(v_{i2})= 1 \text{ if } 1 \leq i \leq n$$

Then this coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(F_n) = \sum_{v \in V(G)} C(v) = n+2$ .

7) *Theorem.2.7.* The Double Wheel graph,  $DW_n, n \geq 4$  and  $n$  is even is proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(DW_n) = n+2$ .

Proof : Let  $v$  be the central vertex. Let  $\{v_1, v_2, v_3, \dots, v_n\}$  and  $\{u_1, u_2, u_3, \dots, u_n\}$  be vertices of inner and outer cycles of  $C_n$  respectively.

Define  $C : V(DW_n) \rightarrow \{0, 1, 2\}$  as follows.

$$C(v) = 2$$

$$C(u_{2i})= 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(u_{2i-1})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i})= 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$



$$C(w_{2i-1})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(DW_n) = \sum_{v \in V(G)} C(v) = n+2$

8) *Theorem. 2.8.* The Crown graph  $C_n^+$  is Proper Roman colourable,  $n \geq 4$  and its Proper Roman chromatic number is given by  $\chi_R$

$$C_n^+ = \begin{cases} 3 \binom{n}{2} & \text{if } n \text{ is even} \\ 3 \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let the vertices on the cycle be  $v_1, v_2, v_3, \dots, v_n$  and the pendent vertices corresponding to the cycle be  $w_1, w_2, w_3, \dots, w_n$ .

a) *Case.1.*  $n \geq 4$  and  $n$  is even

Define  $C : V(C_n^+) \rightarrow \{0,1,2\}$  as follows.

$$C(v_{2i})=2 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(w_{2i-1})=1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(C_n^+) = \sum_{v \in V(G)} C(v) = 3 \binom{n}{2}$ .

b) *Case.2.*  $n \geq 4$  and  $n$  is odd

Define  $C : V(C_n^+) \rightarrow \{0,1,2\}$  as follows.

$$C(v_{2i})=2 \text{ if } 1 \leq i \leq \frac{n-1}{2}$$

$$C(v_{2i-1})=0 \text{ if } 1 \leq i \leq \frac{n-1}{2}$$

$$C(v_n)=1$$

$$C(w_{2i})=0 \text{ if } 1 \leq i \leq \frac{n-1}{2}$$

$$C(w_{2i-1})=1 \text{ if } 1 \leq i \leq \frac{n-1}{2}$$

$$C(w_n)=2$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(C_n^+) = \sum_{v \in V(G)} C(v) = 3 \lfloor \frac{n}{2} \rfloor$ .

9) *Theorem. 2.9.* The Double Crown graph,  $C_n^{++}$ ,  $n > 3$  is Proper Roman colourable and its chromatic number is given by  $\chi_R$

$$C_n^{++} = \begin{cases} 2n & \text{if } n \text{ is even} \\ 2 \lfloor \frac{n}{2} \rfloor + n + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let us label  $v_1, v_2, v_3, \dots, v_n$  as the vertices of the cycle  $C_n$ . Let the pendent edges corresponding to each vertex  $v_i$  be labeled as  $v_{i1}, v_{i2}$

a) *Case.1.*  $n > 3$  and  $n$  is even

Define  $C : V(C_n^{++}) \rightarrow \{0,1,2\}$  as follows.

$$C(v_{2i})=2 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{(2i-1)1})=1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{(2i-1)2})=1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{(2i)1})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{(2i)2})=0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by

$$\chi_R(C_n^{++}) = \sum_{v \in V(G)} C(v) = 2 \frac{n}{2} + n = 2n.$$

b) *Case.2.*  $n > 3$  and  $n$  is odd

Define  $C : V(C_n^{++}) \rightarrow \{0,1,2\}$  as follows.

$$C(v_{2i})=2 \text{ if } 1 \leq i \leq \frac{n-1}{2}$$

$$C(v_{2i-1}) = 0 \text{ if } 1 \leq i \leq \frac{n-1}{2}.$$

$$C(v_n) = 1.$$

$$C(v_{(2i-1)1}) = 1 \text{ if } 1 \leq i \leq \frac{n-1}{2}.$$

$$C(v_{(2i-1)2}) = 1 \text{ if } 1 \leq i \leq \frac{n-1}{2}.$$

$$C(v_{(2i)1}) = 0 \text{ if } 1 \leq i \leq \frac{n-1}{2}.$$

$$C(v_{(2i)2}) = 0 \text{ if } 1 \leq i \leq \frac{n-1}{2}.$$

$$C(v_{n1}) = C(v_{n2}) = 2$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(C_n^{++}) = \sum_{v \in V(G)} C(v) = 2 \left\lceil \frac{n}{2} \right\rceil + n + 2$ .

10) *Theorem. 2.10.* The Web graph,  $Wb_n$ ,  $n \geq 4$  and  $n$  is even is Proper Roman colourable and its Proper Roman chromatic number is given by  $\chi_R(Wb_n) = 2n + 2$

Proof: Let  $n \geq 4$  and  $n$  is even. Let the central vertex of the Web graph,  $Wb_n$  be  $v$ . Let the vertices on the innercycle be  $v_1, v_2, v_3, \dots, v_n$  be the vertices on the outercycle be  $u_1, u_2, u_3, \dots, u_n$  and the pendent vertices be  $w_1, w_2, w_3, \dots, w_n$ .

Define  $C : V(Wb_n) \rightarrow \{0, 1, 2\}$  as follows:

$$C(v) = 2$$

$$C(v_{2i}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(v_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(u_{2i}) = 2 \text{ if } 1 \leq i \leq \frac{n}{2}$$

$$C(u_{2i-1}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

$$C(w_{2i}) = 0 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

$$C(w_{2i-1}) = 1 \text{ if } 1 \leq i \leq \frac{n}{2}.$$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by  $\chi_R(Wb_n) = \sum_{v \in V(G)} C(v) = 2n + 2$ .

### REFERENCES

- [1] J.R.Lewis, Vertex-Edge and Edge-Vertex Parameters in Graphs, Ph.D. Thesis submitted to the Graduate School of Clemson University.
- [2] D. Ochmanek, Time to restructure U.S defence force, Issues in Science & Technology, Winter1996.
- [3] Pagourtzis, P. Penna, K. Schlude, K. Steinhofel, D. S. Taylor and P. Windmayer, "Server placements, Roman domination and other dominating set variants", 2nd IFIP International conference on theoretical Computer Science, Montreal (2002), 280 -291.
- [4] Windmayer, Server placements, Roman domination and other dominating set variants, 2nd IFIP International conference on Theoretical Computer Science, Montreal (2002),280-291.
- [5] Petersen, Defending Roman empire, MathTreck, September 11 (2000),www.maa.org.
- [6] J. Suresh Kumar and Satheesh E.N, Roman labeling of graphs and Application to Military Strategy, International Journal of Mathematics Trends and Technology (IJMTT) – Volume 52 Number 2 December 2017.
- [7] J. Suresh Kumar, Graph Colouring Parameters-A Survey, International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 7 Issue IV, Apr 2019.
- [8] J. Suresh Kumar, Pseudo-Complete Color Critical Graphs, International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 6 Issue I, January 2018.
- [9] J. Suresh Kumar, Diameter and Travers ability of PAN Critical Graphs, International Journal of Mathematics Trends and Technology (IJMTT) – Volume 52 Number 7 December 2017
- [10] J. Suresh Kumar, Degrees and Degree Sequences of PAN Critical Graphs, Int. J. Math. And Appl., 6(1{B})(2018), 1025{1028
- [11] J. Suresh Kumar, Low and High Vertices in Edge Critical Graphs, International Journal of Mathematics Trends and Technology (IJMTT), Volume 65, Issue 4, April, 2019.
- [12] J. Suresh Kumar, Roman colouring of Graphs and Application to Military Strategy, International Journal for Research in Applied Science & Engineering Technology, March 2020.
- [13] J. Suresh Kumar and Preethi K Pillai, Proper Roman coloring of graphs, International Journal of Creative Research Thoughts (IJCRT), Volume.8, Issue.3, March, 2020.
- [14] Frank Harary, Graph Theory, Reading mass, 1969.



10.22214/IJRASET



45.98



IMPACT FACTOR:  
7.129



IMPACT FACTOR:  
7.429



# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24\*7 Support on Whatsapp)