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# Half Logistic NHE: Properties and Application

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***Abstract:** Nadarajah and Haghighi has defined the extension of the exponential distribution we have called this as NHE distribution. It is continuous probability distribution with a wide range of applications such as in life testing experiments, reliability analysis, applied statistics and clinical studies. However, it is not flexible enough for modeling heavily skewed datasets as compared to modified distributions. In this study, we have generated a new continuous distribution having three parameters based on the half logistic-Generating family called half logistic NHE. The structural properties of this model are explored such as the probability density, cumulative density, hazard rate, reversed hazard rate, and quantile functions. The model parameters are estimated using the three well-known methods namely maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. Further, we have computed the Fisher information matrix and asymptotic confidence intervals for ML estimators. The potentiality of the proposed distribution is also evaluated by fitting it in contrast with some other existing distributions using a real life data.*
***Keywords:** Half-logistic distribution, NHE, Estimation, Maximum likelihood.*

## I. INTRODUCTION

In most of the literature of statistics it is found that the study of reliability and survival analysis in various fields of applied statistics and life sciences, the probability distributions are often used. In modeling survival data, existing models do not always present a better fit. Hence most of the researchers are paying attention to generalizing classical distributions and investigating their flexibility and applicability. Usually, these new generalized models provide an improved fit as compared to usual classical distributions and are obtained by introducing one or more additional shape parameter(s) to the baseline distribution.

In probability theory and statistics, the exponential distribution plays a significant role in analyses of survival data. It is the probability distribution of the time between events in a Poisson point process, i.e., a process in which events occur continuously and independently at a constant average rate. It is a particular case of the gamma distribution. It is the continuous analog of the geometric distribution, and it has the key property of being memoryless. In addition to being used for the analysis of Poisson point processes, it is found in various other contexts.

For a few decades, it is found that the exponential distribution is taken as base distribution to generate a new family of distribution. The modifications of the exponential distribution were introduced by different researchers, some of them are, beta exponential (Nadarajah and Kotz, 2006), Gupta and Kundu (2007) have presented the generalized exponential (GE) with some development, Abouammoh & Alshingiti (2009) has introduced the reliability estimation of the generalized inverted exponential distribution, beta GE (Barreto-Souza et al., 2010), Exponential Extension (EE) distribution (Kumar, 2010), KW (Kumaraswamy) exponential (Cordeiro and de Castro, 2011), Nadarajah & Haghighi (2011) have presented an extension of the exponential distribution, gamma EE by (Ristic and Balakrishnan, 2012), Transmuted EE distribution by (Merovci, 2013), Lemonte, A. J. (2013) has introduced a new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Gomez et al. (2014) have presented a new extension of the exponential distribution. The exponentiated exponential geometric (Louzada et al., 2014) and Kumaraswamy transmuted exponential (Afify et al., 2016) distributions. Mahdavi & Kundu (2017) have presented a new method for generating distributions with an application to the exponential distribution.

Recently, the Alpha power transformed extended exponential distribution have introduced by (Hassan et al., 2018). Almarashi et al. (2019) have presented a new extension of exponential distribution with some statistical properties. Abdulkabir & Ipinyomi, (2020) have introduced the Type II half-logistic exponentiated exponential distribution.

Balakrishnan (1985) has developed the half logistic distribution having cumulative distribution function (CDF) and the probability density function (PDF) as

$$F_B(v;\theta)=\frac{1-e^{-\theta v}}{1+e^{-\theta v}}\quad ;\quad v>0,\theta>0$$

and

$$f_B(v;\theta)=\frac{2\theta e^{-\theta v}}{\left(1+e^{-\theta v}\right)^2}\ ;\ v>0,\theta>0$$

Also, the cumulative distribution function (CDF) and the probability density function (PDF) of the type-I half logistic-Generating family having shape parameter λ has introduced by (Cordeiro et al., 2015) are respectively given by

$$F(t;\lambda,\rho)=\frac{1-\left[1-G(t;\rho)\right]^{\lambda}}{1+\left[1-G(t;\rho)\right]^{\lambda}};\quad t,\lambda>0 \tag{1.1}$$

and

$$f(t;\lambda,\rho)=\frac{2\lambda g(t;\rho)\left(1-G(t;\rho)\right)^{\lambda-1}}{\left[1+\left(1-G(t;\rho)\right)^{\lambda}\right]^2},\quad t,\lambda>0. \tag{1.2}$$

where $G(t;\rho)$ and $g(t;\rho)$ are CDF and PDF of baseline distribution, and $\rho$ is the parameter space.

The main goal of this work is to launch a more flexible distribution by inserting just one extra parameter to the NHE distribution (Nadarajah & Haghighi, 2011) to attain a better fit to the real data. We have discussed some distributional properties of the half-logistic NHE distribution and its applicability. The remaining parts of the proposed study are arranged as follows. In Section 2 we present the new half-logistic NHE distribution and its various mathematical and statistical properties. We have employed three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and suitability of the proposed distribution. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and CAIC criterion for ML, LSE, and CVM. Finally, in Section 5 we present some concluding remarks.

## II. THE HALF LOGISTIC NHE DISTRIBUTION

Nadarajah & Haghighi (2011) has defined the extension of the exponential distribution we have called this as NHE distribution. The CDF of NHE is defined as

$$F(x;\alpha,\beta)=1-e^{\{1-(1+\alpha x)^{\beta}\}};\quad \alpha,\beta>0,\ x>0 \tag{2.1}$$

The corresponding PDF can be written as

$$f(x;\alpha,\beta)=\alpha\beta(1+\alpha x)^{\beta-1}e^{\{1-(1+\alpha x)^{\beta}\}};\quad \alpha,\beta>0,\ x>0 \tag{2.2}$$

Substituting (2.1) and (2.2) in (1.1) and (1.2) we get the CDF of half-logistic NHE distribution, which is defined as

$$F(x)=\frac{1-e^{\lambda\{1-(1+\alpha x)^{\beta}\}}}{1+e^{\lambda\{1-(1+\alpha x)^{\beta}\}}}\ ;\alpha,\beta,\lambda>0,\ x>0 \tag{2.3}$$

And the PDF of half-logistic NHE can be expressed as

$$f(x)=\frac{2\alpha\beta\lambda(1+\alpha x)^{\beta-1}e^{\lambda\{1-(1+\alpha x)^{\beta}\}}}{\left[1+e^{\lambda\{1-(1+\alpha x)^{\beta}\}}\right]^2};\alpha,\beta,\lambda>0,\ x>0 \tag{2.4}$$

Survival function is

$$R(x)=\frac{2e^{\lambda\{1-(1+\alpha x)^{\beta}\}}}{1+e^{\lambda\{1-(1+\alpha x)^{\beta}\}}};\alpha,\beta,\lambda>0,\ x>0 \tag{2.5}$$

*A. Hazard Function of Half-logistic NHE*

Let t be survival time of a component or item and we want to calculate the probability that it will not survive for an additional time $\Delta t$ then, hazard rate function is,

$$h(t)=\lim_{\Delta t\to 0}\frac{prob(t\le T<t+\Delta t)}{\Delta t.R(t)}=\frac{f(t)}{R(t)}=\frac{f(t)}{1-F(t)};\ 0<t<\infty$$

where R(t) is a reliability function.

$$h(x)=\frac{2\alpha\beta\lambda(1+\alpha x)^{\beta-1}}{1+e^{\lambda\{1-(1+\alpha x)^{\beta}\}}};\alpha,\beta,\lambda>0,\ x>0 \tag{2.6}$$

*B. Reverse Hazard Function of half-logistic NHE*

The reverse hazard function of half-logistic NHE can be written as

$$h_{rev}(x)=\frac{f(x)}{F(x)}=\frac{2\alpha\beta\lambda(1+\alpha x)^{\beta-1}e^{\lambda\{1-(1+\alpha x)^{\beta}\}}}{1-e^{2\lambda\{1-(1+\alpha x)^{\beta}\}}}\ ;\alpha,\beta,\lambda>0,\ x>0 \tag{2.7}$$

*C. Quantile Function*

Let $X$ be a random variable having a distribution function $F_X(x)$. Let $p\in(0,1)$, the p-quantile of X, denoted by $Q_X(p)$ is

$$Q_X(p)=F_X^{-1}(p)$$

$$Q_X(p)=\frac{1}{\alpha}\left[\left\{1-\frac{1}{\lambda}\ln\left(\frac{1-p}{1+p}\right)\right\}^{1/\beta}-1\right];\ 0<p<1 \tag{2.8}$$

The random deviate generation for the half-logistic NHE can be expressed as,

$$x=\frac{1}{\alpha}\left[\left\{1-\frac{1}{\lambda}\ln\left(\frac{1-u}{1+u}\right)\right\}^{1/\beta}-1\right];\ 0<u<1 \tag{2.9}$$

*D. Skewness and Kurtosis of half-logistic NHE Distribution*

The coefficient of skewness and kurtosis are important measures of dispersion in descriptive statistics. These measures are used mostly in data analysis to study the shape of the distribution or data set. The Bowley's coefficient of skewness based on quartiles is,

$$S_k(Bowley)=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},\ \text{and}$$

Coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_u(Moors)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

Plots of probability density function and hazard rate function of HL-NHE(α,β,λ) with different values of parameters are presented in Figure 1.

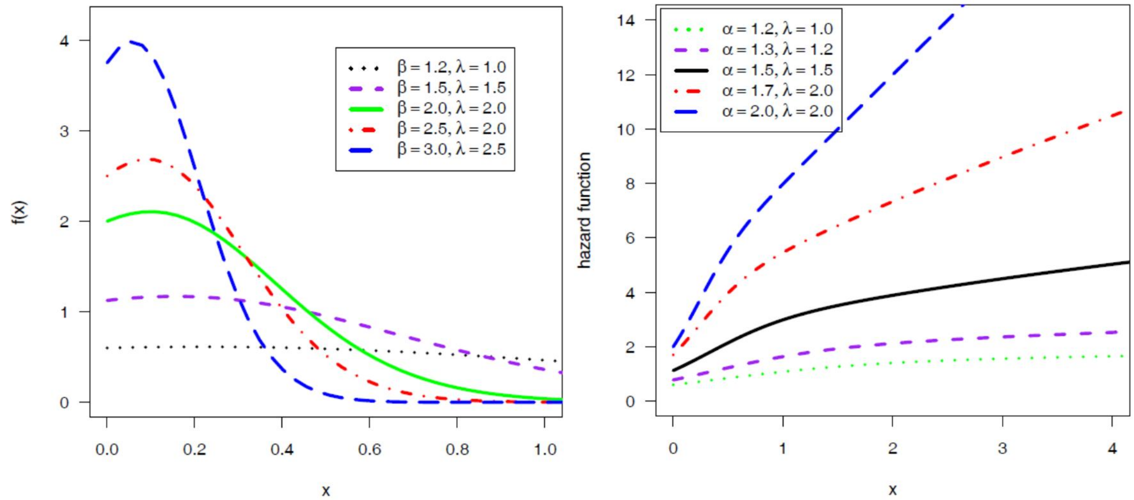


Figure 1. Plots of PDF (left panel) and hazard function (right panel) for fixed α and different values of β and λ.

## III. METHOD OF ESTIMATION

In this section we have presented three methods of estimation of the model parameters, which are as follows,

### *A. Maximum Likelihood Estimation (MLE)*

If $x_1, x_2, ......x_n$ is a random sample from $HL-NHE(\alpha,\beta,\lambda)$ then the likelihood function, $L(\alpha,\beta,\lambda)$ is given by,

$$L(\delta; x_1, x_2......x_n) = g(x_1, x_2, ......x_n / \delta) = \prod_{i=1}^{n} g(x_i / \delta)$$

$$L(\alpha,\beta,\lambda) = 2\alpha\beta\lambda \prod_{i=1}^{n} \frac{(1+\alpha x_i)^{\beta-1} e^{\lambda\{1-(1+\alpha x_i)^{\beta}\}}}{\left[1+e^{\lambda\{1-(1+\alpha x_i)^{\beta}\}}\right]^2}; \quad \alpha,\beta,\lambda > 0,\ x > 0$$

The log-likelihood density is

$$l = n\ln(2\alpha\beta\lambda) + (\beta-1)\sum_{i=1}^{n}\ln(1+\alpha x_i) + \lambda\sum_{i=1}^{n}\left\{1-(1+\alpha x_i)^{\beta}\right\} - 2\sum_{i=1}^{n}\ln\left[1+e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}\right] \qquad (3.1.1)$$

Differentiating (3.1.1) with respect to α, β, and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i} - \beta\lambda\sum_{i=1}^{n} x_i(1+\alpha x_i)^{\beta-1} - 2\beta\lambda\sum_{i=1}^{n}\frac{x_i(1+\alpha x_i)^{\beta-1} e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}}{1+e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(1+\alpha x_i) - \lambda\sum_{i=1}^{n}(1+\alpha x_i)^{\beta}\ln(1+\alpha x_i) + 2\lambda\sum_{i=1}^{n}\frac{(1+\alpha x_i)^{\beta} e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}\ln(1+\alpha x_i)}{1+e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n}(1+\alpha x_i)^{\beta} - 2\sum_{i=1}^{n}\frac{[1-(1+\alpha x_i)^{\beta}]e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}}{1+e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}}$$

 

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Solving non-linear equations $\frac{\partial l}{\partial \alpha}=0,\ \frac{\partial l}{\partial \beta}=0\ and\ \frac{\partial l}{\partial \lambda}=0$, for α, β, and λ, we get the maximum likelihood estimate $\hat{\alpha}, \hat{\beta}\ and\ \hat{\lambda}$ of the parameters α, β, and λ. Maximization of (3.1.1) can be obtained by using computer software like R, Matlab, etc. For the interval estimation of α, β, and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α, β, and λ can be obtained as

$$I=\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} \end{bmatrix}$$

where

$$I_{\alpha\alpha}=-\frac{n}{\alpha^2}-\sum_{i=1}^{n}\frac{(\beta-1)x_i^2}{(1+\alpha x_i)^2}-\beta\lambda(\beta-1)\sum_{i=1}^{n}x_i^2(1+\alpha x_i)^{\beta-2}$$
$$-\sum_{i=1}^{n}\frac{2\beta\lambda x_i(1+\alpha x_i)^{\beta-2}z_i\left[z_i(\beta-1)-\beta\lambda u+\beta-1\right]}{(1+z_i)^2}$$

$$I_{\beta\beta}=-\frac{n}{\beta^2}-\sum_{i=1}^{n}\frac{(\beta-1)x_i^2}{(1+\alpha x_i)^2}-\alpha\beta\sum_{i=1}^{n}(1+\alpha x_i)^{\beta-1}\ln(1+\alpha x_i)$$
$$-\sum_{i=1}^{n}\frac{2\lambda u_i\{\ln(1+\alpha x_i)\}^2 z_i\left[z_i-\lambda u_i+1\right]}{(1+z_i)^2}$$

$$I_{\lambda\lambda}=-\frac{n}{\lambda^2}+\sum_{i=1}^{n}\frac{2(1-u_i)^2 z_i}{(1+z_i)^2}$$

$$I_{\alpha\beta}=\sum_{i=1}^{n}\frac{x_i}{(1+\alpha x_i)}+\lambda\sum_{i=1}^{n}x_i(1+\alpha x_i)^{\beta-1}+\beta\lambda\sum_{i=1}^{n}x_i(1+\alpha x_i)^{\beta-1}\ln(1+\alpha x_i)-\sum_{i=1}^{n}\frac{2\lambda(1+\alpha x_i)^{\beta-1}z_i}{1+z}$$
$$-\sum_{i=1}^{n}\frac{2\beta\lambda z_i(1+\alpha x_i)^{\beta-1}\ln(1+\alpha x_i)}{1+z_i}+\sum_{i=1}^{n}\frac{2\beta\lambda^2 z_i(1+\alpha x_i)^{2\beta-1}\ln(1+\alpha x_i)}{1+z_i}$$
$$-\sum_{i=1}^{n}\frac{2\beta\lambda^2 z_i^2(1+\alpha x_i)^{2\beta-1}\ln(1+\alpha x_i)}{(1+z_i)^2}$$

$$I_{\alpha\lambda}=-\beta\sum_{i=1}^{n}x_i(1+\alpha x_i)^{\beta-1}-\sum_{i=1}^{n}\frac{2\beta(1+\alpha x_i)^{\beta-1}z_i\left[z_i+\lambda(1-u_i)+1\right]}{(1+z_i)^2}$$

$$I_{\beta\lambda}=-\sum_{i=1}^{n}u_i\ln(1+\alpha x_i)+\sum_{i=1}^{n}\frac{2u_i\ln(1+\alpha x_i)z_i\left[z_i+\lambda u_i+\lambda+1\right]}{(1+z_i)^2}$$

where $u_i=(1+\alpha x_i)^{\beta}$ and $z_i=e^{\lambda\left\{1-(1+\alpha x_i)^{\beta}\right\}}$

Let $\Phi = (\alpha, \beta, \lambda)$ denote the parameter space and the corresponding MLE of $\Phi$ as $\hat{\Phi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, then $\left(\hat{\Phi} - \Phi\right) \rightarrow N_3\left[0, \left(I\left(\Phi\right)\right)^{-1}\right]$ follows the asymptotic multivariate normal distribution, where $I\left(\Phi\right)$ is the Fisher's information matrix. By applying the Newton-Raphson algorithm to maximize the likelihood (3.1) produces the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[I\left(\Phi\right)\right]^{-1} = \begin{bmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\beta}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) & \operatorname{cov}(\hat{\beta}, \hat{\lambda}) & \operatorname{var}(\hat{\lambda}) \end{bmatrix} \tag{3.1.2}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β, and λ can be constructed as,

$$\hat{\alpha} \pm z_{\alpha/2} SE(\hat{\alpha}) \text{ , } \hat{\beta} \pm Z_{\alpha/2} SE(\hat{\beta}) \text{ and} \tag{3.5}$$

$$\hat{\lambda} \pm z_{\alpha/2} SE(\hat{\lambda})$$

where $z_{\alpha/2}$ is the upper percentile of standard normal variate.

### *B. Method of Least-Square Estimation (LSE)*

Swain et al. (1988) has introduced the LS and weighted least square estimators to estimate the parameters of Beta distributions. In this article, the same technique is applied for the half logistic NHE distribution. The least-square estimators of the unknown parameters α, β, and λ of half logistic NHE distribution can be obtained by minimizing

$$U\left(X; \alpha, \beta, \lambda\right) = \sum_{j=1}^{n} \left[G(X_j) - \frac{j}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α, β, and λ.

Suppose $G(X_{(j)})$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots\ldots < X_{(n)}$, where $\left\{X_1, X_2, \ldots\ldots, X_n\right\}$ is a random sample of size n from a distribution function *G(.)*. Therefore, in this case, the least square estimators of α, β, and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ respectively, can be obtained by substituting (2.3) in (3.2.1) and minimizing

$$U\left(X; \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right) = \sum_{j=1}^{n} \left[\frac{1 - e^{\lambda\{1-(1+\alpha x_j)^{\beta}\}}}{1 + e^{\lambda\{1-(1+\alpha x_j)^{\beta}\}}} - \frac{j}{n+1}\right]^2 \tag{3.2.2}$$

with respect to α, β, and λ.

$$\frac{\partial U}{\partial \alpha} = 4\beta\lambda \sum_{j=1}^{n} \left[\frac{1 - v(x_j)}{1 + v(x_j)} - \frac{j}{n+1}\right] \left[\frac{x_j (1+\alpha x_j)^{\beta-1} v(x_j)}{\left\{1 + v(x_j)\right\}^2}\right]$$

$$\frac{\partial U}{\partial \beta} = 4\lambda \sum_{j=1}^{n} \left[\frac{1 - v(x_j)}{1 + v(x_j)} - \frac{j}{n+1}\right] \left[\frac{(1+\alpha x_j)^{\beta} v(x_j) \ln(1+\alpha x_j)}{\left\{1 + v(x_j)\right\}^2}\right]$$

$$\frac{\partial U}{\partial \lambda} = -4 \sum_{j=1}^{n} \left[\frac{1 - v(x_j)}{1 + v(x_j)} - \frac{j}{n+1}\right] \left[\frac{\left\{1 - (1+\alpha x_j)^{\beta}\right\} v(x_j)}{\left\{1 + v(x_j)\right\}^2}\right]$$

where $v(x_j) = e^{\lambda\{1-(1+\alpha x_j)^\beta\}}$

The weighted least square estimators of the unknown parameters can be obtained by minimizing

$$W(X;\alpha,\beta,\lambda) = \sum_{j=1}^{n} w_j \left[ G(X_{(j)}) - \frac{j}{n+1} \right]^2$$

with respect to α, β, and λ. The weights $w_j$ are $w_j = \frac{1}{V(X_{(j)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$

Hence, the weighted least square estimators of α, β, and λ respectively, can be obtained by minimizing,

$$W(X;\alpha,\beta,\lambda) = (n+1)^2(n+2)\sum_{j=1}^{n} \frac{1}{j(n-j+1)} \left[ \frac{1-e^{\lambda\{1-(1+\alpha x_j)^\beta\}}}{1+e^{\lambda\{1-(1+\alpha x_j)^\beta\}}} - \frac{j}{n+1} \right]^2 \qquad (3.2.3)$$

with respect to α, β, and λ.

*C. Method of Cramer-Von-Mises (CVM)*

Another method of estimation is Cramér-von-Mises type minimum distance estimation, (Macdonald 1971) and it provides the experimental evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α, β, and λ are obtained by minimizing the function

$$C(\alpha,\beta,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n} \left[ F(x_{i:n} \mid \alpha,\beta,\lambda) - \frac{2i-1}{2n} \right]^2 \qquad (3.3.1)$$

$$= \frac{1}{12n} + \sum_{i=1}^{n} \left[ \frac{1-e^{\lambda\{1-(1+\alpha x_i)^\beta\}}}{1+e^{\lambda\{1-(1+\alpha x_i)^\beta\}}} - \frac{2i-1}{2n} \right]^2$$

Thus CVM estimators of α, β, and λ are obtained by solving the following equations simultaneously,

$$\frac{\partial C(\alpha,\beta,\lambda)}{\partial \alpha} = 0, \frac{\partial C(\alpha,\beta,\lambda)}{\partial \beta} = 0 \text{ and } \frac{\partial C(\alpha,\beta,\lambda)}{\partial \lambda} = 0 \text{ where}$$

$$\frac{\partial C(\alpha,\beta,\lambda)}{\partial \alpha} = \frac{1}{12n} + 4\beta\lambda \sum_{i=1}^{n} \left[ \frac{1-B(x_i)}{1+B(x_i)} - \frac{2i-1}{2n} \right] \left[ \frac{x_i(1+\alpha x_i)^{\beta-1} B(x_i)}{\{1+B(x_i)\}^2} \right]$$

$$\frac{\partial C(\alpha,\beta,\lambda)}{\partial \beta} = \frac{1}{12n} + 4\lambda \sum_{i=1}^{n} \left[ \frac{1-B(x_i)}{1+B(x_i)} - \frac{2i-1}{2n} \right] \left[ \frac{(1+\alpha x_i)^{\beta} B(x_i) \ln(1+\alpha x_i)}{\{1+B(x_i)\}^2} \right]$$

$$\frac{\partial C(\alpha,\beta,\lambda)}{\partial \lambda} = \frac{1}{12n} - 4 \sum_{i=1}^{n} \left[ \frac{1-B(x_i)}{1+B(x_i)} - \frac{2i-1}{2n} \right] \left[ \frac{\{1-(1+\alpha x_i)^{\beta}\} B(x_i)}{\{1+B(x_i)\}^2} \right]$$

where $B(x_i) = e^{\lambda\{1-(1+\alpha x_i)^\beta\}}$.

## IV. APPLICATIONS TO REALISTIC DATA SETS

In this section, we demonstrate the applicability of half logistic NHE distribution using a real dataset used by earlier researchers. The following data represent the service times of 63 aircraft wind shield Kundu & Raqab (2009) and listed as follows:

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719,2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900,2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665,1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543,5.140

We have calculated the MLEs directly by using optim() function (Schmuller, 2017) in R software (R Core Team, 2020) and (Rizzo, 2008) by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha}$ = 0.1649, $\hat{\beta}$ = 3.7152, $\hat{\lambda}$= 0.5881 and corresponding value of Log-Likelihood value is -98.09901. In Table 1 we have presented the MLE's with their standard errors (SE) for α, β, and λ.

Table 1
MLE and SE for α, β, and λ

| Parameter | MLE | SE |
|---|---|---|
| alpha | 0.1649 | 0.7653 |
| beta | 3.7152 | 9.5424 |
| lambda | 0.5881 | 1.4300 |

An estimate of the variance-covariance matrix by using MLEs, using equation (3.1.2) is

$$\begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 0.5856 & -7.2774 & -1.0766 \\ -7.2774 & 91.0568 & 13.1825 \\ -1.0766 & 13.1825 & 2.0450 \end{pmatrix}$$

In Figure 2 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β and λ. We have found that ML estimates of α, β, and λ exist and can be obtained uniquely.

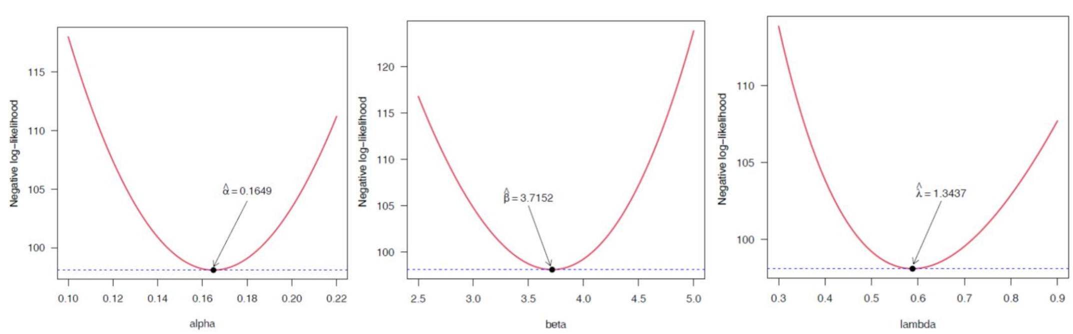


Figure 2. The plots of the profile log-likelihood function of ML estimates of α, β, and λ.

For the test of goodness of fit and adequacy of the proposed model, Akaike information criterion (AIC), Bayesian information criterion (BIC) and Corrected Akaike information criterion (AICC) are calculated for the MLE, LSE and CVM estimators and presented in Table 2. It is observed that MLEs are superior to LSE and CVM.

Table 2
Estimated parameters, log-likelihood, AIC, BIC and AICC

| Method | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | -LL | AIC | BIC | AICC |
|---|---|---|---|---|---|---|---|
| MLE | 0.1650 | 3.7152 | 0.5881 | 98.0990 | 202.198 | 208.6274 | 202.5915 |
| LSE | 0.2672 | 3.0175 | 0.4162 | 98.1202 | 202.2404 | 208.6698 | 202.6338 |
| CVE | 0.3257 | 2.8967 | 0.3346 | 98.2193 | 202.4385 | 208.8679 | 202.832 |

To evaluate the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit. From Figure 3 we have shown that the HL-NHE model fits the data very well.

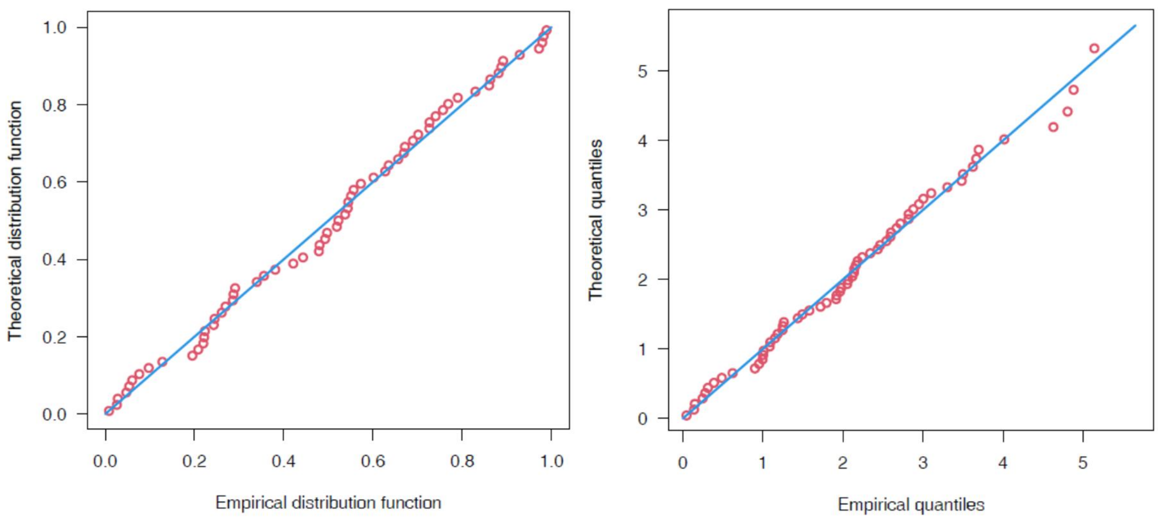


Figure 3. The P-P plot (left panel) and Q-Q plot (right panel) of HL-NHE distribution

We have considered some alternatives distributions for the comparison of goodness of fit and flexibility of the observed distribution are as follows.

### A. *Generalized Gompertz (GG) Distribution*

The pdf of GG distribution (El-Gohary, 2013) is

$$f_{GG}(x) = \theta\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)} \left[1-e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\right]^{\theta-1}; \quad \lambda\theta > 0, \alpha \ge 0, x \ge 0$$

### B. *Generalized Exponential Extension (GEE) Distribution*

The PDF of generalized exponential extension distribution (Lemonte, 2013) with parameters $\alpha$ , $\beta$ and $\lambda$ is

$$f_{GEE}(x;\alpha,\beta,\lambda) = \alpha\beta\lambda\,(1+\lambda x)^{\alpha-1}\, exp\left\{1-(1+\lambda x)^{\alpha}\right\}$$

$$\left[1- exp\left\{1-(1+\lambda x)^{\alpha}\right\}\right]^{\beta-1} \quad ; \; x \ge 0.$$

### C. *Exponential Extension (EE) Distribution*

The density of exponential extension (EE) distribution (Nadarajah & Haghighi, 2011) with parameters α and λ is

$$f_{EE}(x) = \alpha\,\lambda\,(1+\lambda x)^{\alpha-1}\,\exp\left\{1-(1+\lambda x)^{\alpha}\right\} \quad ; x \ge 0, \alpha > 0, \lambda > 0.$$

### D. *Weibull Distribution*

The probability density function of Weibull (W) distribution is

$$f_W(x) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}}; \; \lambda\theta > 0, x \ge 0$$

*E. Exponentiated Exponential Poisson (EEP)*

The probability density function of EEP (Ristić & Nadarajah, 2014) can be expressed as

$$f\left(x\right)=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x}\right)^{\alpha}\right\}\quad ; x>0,\alpha>0,\lambda>0$$

For the assessment of potentiality of the proposed model we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) and these are presented in Table 2.

Table 2
Log-likelihood (LL), AIC, BIC, CAIC and HQIC

| Model | -LL | AIC | BIC | CAIC | HQIC |
|---|---|---|---|---|---|
| HLNHE | 98.0990 | 202.1980 | 208.6274 | 202.6048 | 204.7267 |
| GG | 98.2316 | 202.4633 | 208.8927 | 202.8701 | 204.9920 |
| GEE | 98.6627 | 203.3254 | 209.7548 | 203.7322 | 205.8541 |
| EE | 100.1167 | 204.2335 | 208.5197 | 204.4270 | 205.9193 |
| Weibull | 100.3177 | 204.6354 | 208.9217 | 204.8354 | 206.3212 |
| EEP | 103.5468 | 213.0936 | 219.5230 | 213.5004 | 215.6224 |

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of HLNHE, generalized Gompertz (GG), generalized exponential extension (GEE), exponential extension (EE), Weibull and EEP distributions are presented in Figure 4.

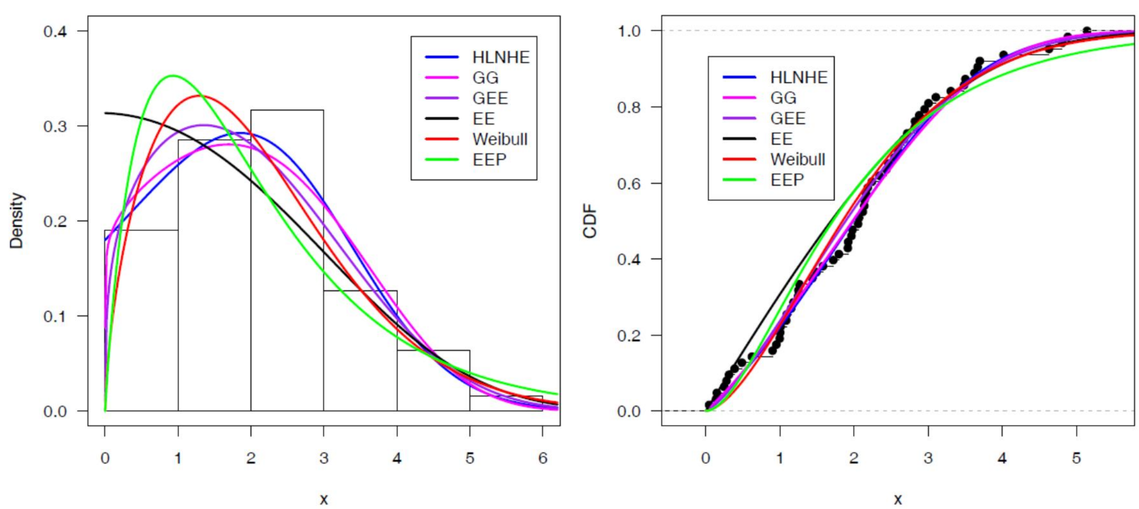


Figure 4. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the HLNHE distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. From Table 4 the result shows that the HLNHE distribution has the minimum value of the test statistic and higher *p*-value hence we conclude that the HLNHE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 4
The goodness-of-fit statistics and their corresponding p-value

| Model | *KS(p-value)* | *AD(p-value)* | *CVM(p-value)* |
|---|---|---|---|
| HLNHE | 0.0658(0.9310) | 0.0329(0.9666) | 0.2332(0.9785) |
| GG | 0.0694(0.9009) | 0.0428(0.9197) | 0.2890(0.9460) |
| GEE | 0.0958(0.5771) | 0.0684(0.7633) | 0.4040(0.8442) |
| EE | 0.1446(0.1296) | 0.2949(0.1397) | 1.3923(0.2044) |
| Weibull | 0.1087(0.4168) | 0.0929(0.6222) | 0.6426(0.6080) |
| EEP | 0.1431(0.1373) | 0.2298(0.2167) | 1.3052(0.2306) |

## V. CONCLUSION

In this article, we have introduced the three-parameter half logistic NHE distribution. We have provided the mathematical and statistical properties such as reliability function, hazard function, quantile function, skewness, and kurtosis of the model. Considering a real data set, we have explored the maximum likelihood estimates of the parameters and their corresponding confidence interval. Further other two well-known estimation methods namely least square estimates (LSE) and Cramer-Von-Mises (CVM) methods are employed to estimate the parameters of the proposed model. It is observed that MLEs are quite better than LSE and CVM methods. The comparison is done based on various information criteria such as AIC, BIC, CAIC, HQIC, and Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics, and found that the proposed model is better as compared to generalized Gompertz (GG), generalized exponential extension (GEE), exponential extension (EE), Weibull and EEP distributions. We hope that this probability distribution may be an alternative in the field of survival analysis, probability distribution and applied statistics.

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