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Fundamental Study of Dirac Delta Function

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Abstract: Basic properties of Dirac Delta Function are explored. Inter-relationships among various representations are established. Fourier Integral is achieved from this improper function. The value of Fermi Golden Integral emerge out in a simple manner.
Keywords: Dirac Delta, Fermi Golden Integral, improper function, various representations

I. INTRODUCTION

Dirac delta function has its root in quantum mechanics as a fundamental tool to normalize a plane wave solution. In case of discrete spectrum Kronecker delta is sufficient to ensure the orthonormality and completeness, but for the regime of continuous domain necessity of Dirac delta function comes in to the play. Now we start with defining Kronecker delta function [1]

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \ldots(1)$$

It has a simple property as reflecting the completeness of the spectrum of Eigen values

$$\sum_{i=-\infty}^{i=\infty} \delta_{ij} = 1 \quad , \quad \forall i, j \in \{0,1,2,\ldots\} \qquad \ldots(2)$$

This simple property can be used to transform the argument of any function as given below

$$\sum_{i=-\infty}^{i=\infty} \delta_{ij} f(i) = f(j) \qquad \ldots(3)$$

The Eq. (1) appears in case of Dot Product of vectors, partial differentiation of independent coordinates, integration of trigonometric functions, integration of exponential function with imaginary argument etc. Thus we can write obtain an integral representation of Kronecker delta as follows [2]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} d\theta = \begin{Bmatrix} 1, & n = 0 \\ 0, & n \neq 0 \end{Bmatrix} = \delta_{n,0} \Leftrightarrow \delta_{p,q} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(p-q)\theta} d\theta \qquad \ldots(4)$$

The transition from discrete to continuous domain can be accomplished with the substitution$x = \frac{2\pi n}{L}, L \to \infty$. It is worthwhile to mention that n takes values in unit steps but x is continuous due to L.

Now we introduce Dirac delta function in the following manner

$$\delta(x) = \lim_{\Delta x \to 0} \frac{\delta_{n,0}}{\Delta x} \qquad \ldots(5)$$

This definition suggest the constraint that

$$\int dx\, \delta(x) = \sum \lim_{\Delta x \to 0} \frac{\delta_{n,0}}{\Delta x} \Delta x = \sum \delta_{n,0} = 1 \qquad \ldots(6)$$

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The definition given by Eq. (5) reveals that Dirac delta function is not a proper function in the vicinity of origin and vanishes away from it. But the constraint Eq. (6) controls its improperness up to some extent and makes it as a pathological function [3].

$$\delta(x) = \begin{cases} \infty, x = 0 \\ 0, x \neq 0 \end{cases} \qquad \ldots(7)$$

Eq. (7) gives the improper character and to control the behavior up to some extent, we may say that Dirac delta should satisfy the following property

$$x\delta(x) = 0, \qquad \delta'(x) = -\delta(x)/x, \qquad \ldots(8)$$

Above Eq. (8) shows that the slope of it is infinite at the origin due to improper character of the function. Moreover, Eq. (7) reveals that Dirac delta function is an even function while Eq. (8) says its derivative is an odd function.

$$\int_{-\infty}^{\infty} \delta'(x)dx = 0 \qquad \ldots(9)$$

$$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x)dx = -f'(0) \qquad \ldots(10)$$

The n^th derivative is found to be

$$\delta^{(n)}(x) = \frac{(-1)^n n!}{x^n}\delta(x) = \left(-x^{-1}\right)^n \Gamma(n+1)\delta(x) \qquad \ldots(11)$$

Eq. (11) suggests that fractional derivative of Dirac delta function lies in imaginary regime.

Eq. (8) also shows the change of scale property

$$(x-a)\delta(x-a) = 0, \qquad x\delta(x-a) = a\delta(x-a) \qquad \ldots(12)$$

Eq. (12) suggests that Dirac delta function focuses the function at the origin or point of vanishing argument [4].

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad \ldots(13)$$

II. VARIOUS REPRESENTATIONS OF DIRAC DELTA FUNCTION AND INTERRELATIONSHIP-

A. *Plane wave representation*

$$\delta(x) = \lim_{\Delta x \to 0} \frac{\delta_{n,0}}{\Delta x} = \lim_{L \to \infty} \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\varphi} d\varphi}{2\pi/L} = \lim_{L \to \infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(i\left(\frac{2\pi n}{L}\right)\left(\frac{L}{2\pi}\varphi\right)\right) d\left(\frac{L}{2\pi}\varphi\right)$$

$$= \lim_{L \to \infty} \frac{1}{2\pi}\int_{-L/2}^{L/2} \exp(i(x)(k))d(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)dk \qquad \ldots(14).$$

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B. Sinc(x) representation

$$\delta(x)=\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L/2}^{L/2}\exp(i(x)(k))d(k)=\lim_{g\to\infty}\frac{1}{2\pi}\int_{-g}^{g}dx\ \exp(ikx)$$

$$=\lim_{g\to\infty}\frac{\sin gx}{\pi x} \quad \ldots(15)$$

C. Lorentzian Representation

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dk=\lim_{\in\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\in|k|}dk$$

$$\delta(x)=\lim_{\in\to 0+}\frac{1}{2\pi}\int_{-\infty}^{0}\exp(ikx+\in k)dk+\lim_{\in\to 0+}\frac{1}{2\pi}\int_{0}^{\infty}\exp(ikx-\in k)dk$$

$$=\lim_{\in\to 0+}\frac{1}{2\pi i}\left\{\frac{1}{x-i\in}-\frac{1}{x+i\in}\right\}=\lim_{\in\to 0+}\frac{\in}{\pi(x^2+\in^2)}$$

$$\ldots(16)$$

D. Gaussian Representation

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dk=\lim_{\in\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[ikx-\left(\frac{\in k}{2}\right)^2\right]dk$$

$$=\lim_{\in\to 0+}\frac{\exp\left[-(x/\in)^2\right]}{\in\pi}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{\in k}{2}+\frac{x}{i\in}\right)^2\right]d\left(\frac{\in k}{2}+\frac{x}{i\in}\right)$$

$$=\lim_{\in\to 0+}\frac{\exp\left[-(x/\in)^2\right]}{\in\pi}\int_{-\infty}^{\infty}\exp\left[-\xi^2\right]d\xi=\lim_{\in\to 0+}\frac{\exp\left[-(x/\in)^2\right]}{\in\pi}I$$

$$\int_{-\infty}^{\infty}\delta(x)dx=1\Rightarrow I\int_{-\infty}^{\infty}\frac{\exp\left[-(x/\in)^2\right]}{\pi}d\left(\frac{x}{\in}\right)=\frac{I}{\pi}\int_{-\infty}^{\infty}\exp\left[-\xi^2\right]d\xi=1\Rightarrow I=\sqrt{\pi}$$

$$\delta(x)=\lim_{\in\to 0+}\frac{\exp\left[-(x/\in)^2\right]}{\in\sqrt{\pi}}$$

$$\ldots(17)$$

Here it should be emphasized that the value of Γ(½) emerged out by itself.

E. Derivative of Lorentz Heaviside step Function

$$\int_{-\infty}^{x}\delta(x)dx=\int_{-\infty}^{x}\lim_{\in\to 0+}\frac{\in}{\pi(x^2+\in^2)}dx=\lim_{\in\to 0+}\frac{\in}{\pi}\int_{-\infty}^{x}\frac{1}{x^2+\in^2}dx$$

$$=\frac{1}{2}+\lim_{\in\to 0+}\frac{1}{\pi}\arctan\frac{x}{\in}$$

$$sign(x)=\frac{|x|}{x}=\begin{cases}1, & x>0\\ -1, & x<0\end{cases} \qquad \theta(x)=\frac{1+sign(x)}{2}=\begin{cases}1, & x>0\\ 0, & x<0\end{cases}$$

$$|x|=sign(x)\ x\Rightarrow\frac{d}{dx}|x|=sign(x) \quad , \theta(\pm x)=\frac{1\pm sign(x)}{2} \quad , \quad sign(x)=\theta(x)-\theta(-x)$$

$$\int_{-\infty}^{x}\delta(x)dx=\frac{1}{2}+\underset{\in\to 0+}{Lim}\frac{1}{\pi}\arctan\left\{\frac{|x|sign(x)}{\in}\right\}=\frac{1}{2}+sign(x)\underset{\in\to 0+}{Lim}\frac{1}{\pi}\arctan\left\{\frac{|x|}{\in}\right\}$$

$$=\frac{1}{2}+sign(x)\frac{1}{\pi}\arctan\left\{\frac{|x|}{0+}\right\}=\frac{1+sign(x)}{2}=\theta(x)$$

$$\delta(x)=\frac{d\theta(x)}{dx} \qquad \ldots(18)$$

$$\delta(x)=\frac{d}{dx}\left\{\frac{1+sign(x)}{2}\right\}=\frac{1}{2}\frac{d}{dx}sign(x)=\frac{1}{2}\frac{d}{dx}\frac{d|x|}{dx}=\frac{1}{2}\frac{d^2}{dx^2}|x|=\delta(-x) \qquad \ldots(19)$$

$$\delta(ax)=\frac{1}{2}\frac{d^2}{d(ax)^2}|ax|=\frac{1}{2|a|^2}\frac{d^2}{dx^2}|ax|=\frac{1}{|a|}\delta(x) \qquad \ldots(20)$$

F. Dirac Delta Decomposition

$$\delta(x)=\underset{\in\to 0+}{Lim}\frac{1}{2\pi i}\left\{\frac{1}{x-i\in}-\frac{1}{x+i\in}\right\}=\frac{1}{2\pi i}\underset{\in\to 0+}{Lim}\frac{1}{x-i\in}-\frac{1}{2\pi i}\underset{\in\to 0+}{Lim}\frac{1}{x+i\in}$$

$$=\tilde{\delta}^{-}(x)+\tilde{\delta}^{+}(x)=\tilde{\delta}^{+}(x)+\left[\overline{\tilde{\delta}^{+}(x)}\right]=2\operatorname{Re}\tilde{\delta}^{+}(x)=\frac{1}{2\pi i}\underset{\in\to 0+}{Lim}\left[\delta^{-}(x)-\delta^{+}(x)\right]$$

$$\tilde{\delta}^{+}(x)=\frac{1}{2\pi}\underset{\in\to 0+}{Lim}\frac{-1}{\in -ix}=\frac{-1}{2\pi}\underset{\in\to 0+}{Lim}\int_{0}^{\infty}dk\exp[-k(\in -ix)]$$

$$=\frac{-1}{2\pi}\int_{-\infty}^{\infty}dk\ \theta(k)e^{ikx} \qquad \ldots(21)$$

$$\tilde{\delta}^{-}(x)=\frac{1}{2\pi}\underset{\in\to 0+}{Lim}\frac{1}{\in +ix}=\frac{1}{2\pi}\underset{\in\to 0+}{Lim}\int_{0}^{\infty}dk\exp[-k(\in +ix)]$$

$$=\frac{-1}{2\pi}\int_{0}^{-\infty}dk\ e^{ikx}=\frac{1}{2\pi}\int_{-\infty}^{0}dk\ e^{ikx}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\ \theta(-k)\ e^{ikx} \qquad \ldots(22)$$

$$\delta^{\pm}(x)=\frac{1}{x\pm i0}=\frac{x\mp i0}{x^2+0^2}=\frac{x}{x^2+0^2}\mp i\frac{0}{x^2+0^2}=P\frac{1}{x}\mp i\pi\delta(x) \qquad \ldots(23)$$

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Where P stands for Cauchy 's Principal value of the integral. Eq. (23) is known as *Dirac identity*.

$$\delta^{\pm}(x)=\frac{1}{\pm i}\int_{-\infty}^{\infty} dk\,\theta(\pm k)\exp(ikx) \qquad \ldots(24)$$

Eq. (24) is direct manifestation of Eq. (22) and Eq.(23). Eq. (24) can be utilized to obtain further expression of Dirac Delta Function.

$$\delta^{+}(x)=\lim_{\in\to 0}\frac{1}{x+i\in}=\frac{1}{i}\int_{-\infty}^{\infty} dk\,\theta(k)\exp(ikx)=\frac{1}{i}\lim_{g\to\infty}\int_{0}^{g} dk\ \exp(ikx)$$

$$\Rightarrow \lim_{\in\to 0}\frac{x}{x^2+\in^2}-i\lim_{\in\to 0}\frac{\in}{x^2+\in^2}=\lim_{g\to\infty}\frac{1-e^{igx}}{x}=\lim_{g\to\infty}\frac{1-\cos gx}{x}-i\lim_{g\to\infty}\frac{\sin gx}{x}$$

Equating real and imaginary parts , we get

$$\lim_{\in\to 0}\frac{\in}{\pi\left(x^2+\in^2\right)}=\lim_{g\to\infty}\frac{\sin gx}{\pi x}\Rightarrow \delta(x)=\lim_{g\to\infty}\frac{\sin gx}{\pi x} \qquad \ldots(25)$$

$$\lim_{\in\to 0}\frac{x}{x^2+\in^2}=\lim_{g\to\infty}\frac{1-\cos gx}{x}\Rightarrow \delta(x)=\lim_{g\to\infty}\frac{1-\cos gx}{\pi g x^2} \qquad \ldots(26)$$

In obtaining Eq. (26) we noticed that $\boxed{g = 1/\in}$

Eq. (26) can be further molded to get other representation as follows

$$\delta(x)=\lim_{g\to\infty}\frac{1-\cos gx}{\pi g x^2}=\lim_{g\to\infty}\frac{1-\cos 2gx}{2\pi g x^2}=\lim_{g\to\infty}\frac{\sin^2 gx}{\pi g x^2} \qquad \ldots(27)$$

G. Hyperbolic Secant Representation

Dirac Delta function is an improper function and we can develop a proper function which in limiting case produces Dirac delta function.

$$\lim_{\in\to 0}\Delta_{\in}(x)=\delta(x) \qquad \int dx\Delta_{\in}(x)=1$$

$$\delta(x)=\lim_{\in\to 0+}\frac{\exp\left[-(x/\in)^2\right]}{\in\sqrt{\pi}}$$

$$\exp\left\{-(x/\in)^2\right\}\approx 1-(x/\in)^2$$

$$\cosh(x/\in)\approx 1+\tfrac{1}{2}(x/\in)^2$$

$$\sec h^2(x/\in)\approx\left[1+\tfrac{1}{2}(x/\in)^2\right]^{-2}=1-(x/\in)^2\approx\exp\left\{-(x/\in)^2\right\}$$

Hence, we choose

$$\Delta_{\in}(x)=\frac{\sec h^2(x/\in)}{2\in}$$

$$\int dx\frac{\sec h^2(x/\in)}{2\in}=\frac{1}{2}\int d\left(\frac{x}{\in}\right)\sec h^2\left(\frac{x}{\in}\right)=\frac{1}{2}\tanh\left(\frac{x}{\in}\right)\Bigg|_{-\infty}^{\infty}=1$$

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Therefore, we may take squared hyperbolic secant representation in the following manner.

$$\delta(x)=\lim_{\in\to 0}\frac{\sec h^2(x/\in)}{2\in}=\lim_{g\to\infty}\frac{g}{1+\cosh 2gx}=\lim_{g\to\infty}\frac{g/2}{(1+\cosh gx)} \quad \ldots(28)$$

$$\exp\left\{-(x/\in)^2\right\}\approx 1-(x/\in)^2$$

$$\sec h(x/\in)\approx 1-\tfrac{1}{2}(x/\in)^2$$

$$\sec h\left(\sqrt{2}x/\in\right)\approx\left[1-\tfrac{1}{2}\left(\sqrt{2}x/\in\right)^2\right]=1-(x/\in)^2\approx\exp\left\{-(x/\in)^2\right\}$$

$$\delta(x)=\lim_{\in\to 0}\frac{\sqrt{2}}{\pi\in}\sec h\left(\frac{\sqrt{2}x}{\in}\right) \quad \ldots(29)$$

Eq. (16) , (17) and (29) are used to represent a line shape function of a Laser pulse.

III. EVALUATION OF VARIOUS INTEGRALS

A. *Integral of Sinc (x)*

Eq. (25) and Constraint Eq. (6) suggest that

$$\lim_{g\to\infty}\int_{-\infty}^{\infty}dx\frac{\sin gx}{x}=\pi\Rightarrow\lim_{g\to\infty}\int_{-\infty}^{\infty}dgx\frac{\sin gx}{gx}=\pi\Rightarrow\int_{0}^{\infty}dy\frac{\sin y}{y}=\frac{\pi}{2} \quad \ldots(30)$$

B. *Fermi-Golden Integral*

Eq. (26) and Constraint Eq. (6) suggest that

$$\lim_{g\to\infty}\int_{-\infty}^{\infty}dx\frac{\sin^2 gx}{\pi gx^2}=1\Rightarrow\lim_{g\to\infty}\int_{-\infty}^{\infty}d(gx)\frac{\sin^2(gx)}{\pi(gx)^2}=1\Rightarrow\int_{-\infty}^{\infty}dy\frac{\sin^2 y}{y^2}=\pi \quad \ldots(31)$$

This integral is what we exactly need in time dependent perturbation theory in quantum mechanics.

C. *Fourier Integral and Fourier Transforms*

Eq. (3) in case of continuous distribution acquires the form given below.

$$\int dx\, f(x)\delta(x-x')=f(x') \quad \ldots(32)$$

Using Plane wave representation for Dirac Delta function, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dx\;f(x)\exp\left[ik(x-x')\right]=f(x') \quad \ldots(33)$$

Eq. (33) is well- known *Fourier Integral*.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\;\exp\left[-ikx'\right]\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\;f(x)\exp\left[ikx\right]\right\}=f(x')$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\;\exp\left[-ikx'\right]\;F\left[f(x)|\,k\right]=f(x') \quad \ldots(34)$$

$$F\left[f(x)|\,k\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\;\exp\left[ikx\right]\;f(x)$$

These are usual Fourier transforms.

IV. MOST COMMON PROPERTIES

This section is included due to different methods to evaluate some properties of Dirac Delta function.

A. Delta Orthogonality

$$f(x)\delta(x-a)=f(a)\delta(x-a)$$
$$\Rightarrow \delta(a-x)f(x)=f(a)\delta(a-x)\Rightarrow \delta(a-x)\delta(x-b)=\delta(a-b)\delta(a-x)$$
$$\int_{-\infty}^{\infty} dx\delta(a-x)\delta(x-b)=\delta(a-b) \quad \text{...(35)}$$

B. Delta Projection at Roots of the Argument

We define an auxiliary function χ in the following way

$$\chi(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases} \quad , \quad \chi(ax)=\chi(x)\,, \quad a\neq 0 \quad \text{...(36)}$$
$$\chi(x)\delta(x)=\delta(x) \quad \text{...(37)}$$

If function is f(x) as a polynomial of degree n.

$$f(x)=\prod_{i=1}^{n}(x-a_i)=\sum_{i=1}^{n}\chi(x-a_i)\{(x-a_i)f_i(x)\} \quad ,$$
$$f'(x)=\sum_{i=1}^{n}f_i(x), \qquad f_i(a_j)=\delta_{ij}f'(a_i) \quad , \qquad \chi(f)=\sum_{i=1}^{n}\chi(x-a_i)$$
$$\delta(f)=\chi(f)\delta(f)=\sum_{i=1}^{n}\chi(x-a_i)\delta(f)=\sum_{i=1}^{n}\chi(x-a_i)\delta\left[\prod_{j=1}^{n}(x-a_j)\right]$$
$$=\sum_{i=1}^{n}\chi(x-a_i)\delta[(x-a_i)f_i(x)]=\sum_{i=1}^{n}\delta[(x-a_i)f_i(a_i)]=\sum_{i=1}^{n}\frac{\delta(x-a_i)}{|f_i(a_i)|}=\sum_{i=1}^{n}\frac{\delta(x-a_i)}{|f'(a_i)|} \quad \text{...(38)}$$

V. CONCLUSION

Dirac Delta function is widely studied by a number of scholars, but in this paper its usage to find value of Γ(½), to define Fourier Transforms and to evaluate various types of integrals is highlighted. The basic finding is we have to investigate proper function $\Delta_\varepsilon(x)$which in limiting case produces improper function δ(x), then the constraint condition will allow us to evaluate a difficult integral. This makes Dirac Delta function a very powerful tool in Mathematical and theoretical Physics.

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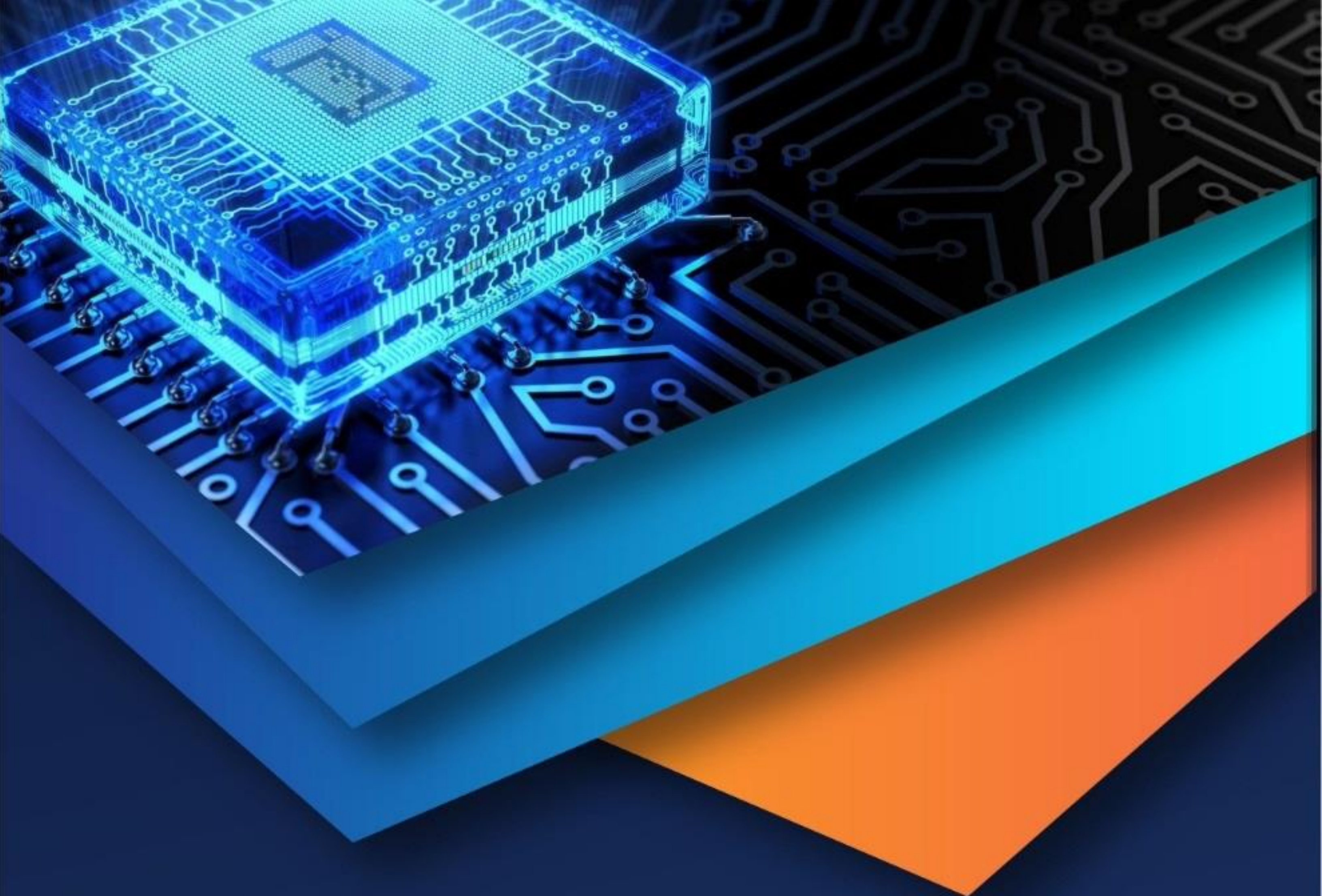

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