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# Iterative Methods for the Solution of Semi-Nonlinear Systems with Linear Diagonals

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**Abstract:** We discuss Jacobi, Gauss-seidel and SOR methods for the solution of semi-nonlinear systems with linear diagonals in this paper. A Comparison of these methods is done through an example.

**Keywords:** Non-linear equations, Iterative methods, Jacobi, Gauss-seidel, SOR.

## I. INTRODUCTION

Let us consider a system of n non-linear equations in n unknowns of the form

$$\left. \begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0 \\ f_2(x_1, x_2, \dots, x_n) &= 0 \\ \cdot & \\ \cdot & \\ f_n(x_1, x_2, \dots, x_n) &= 0 \end{aligned} \right\} \dots\dots\dots(1.1)$$

If one can express the system(1.1) as in the following form

$$\left. \begin{aligned} a_{11}f_{11}(x_1) + a_{12}f_{12}(x_2) + \dots\dots\dots + a_{1n}f_{1n}(x_n) &= b_1 \\ a_{21}f_{21}(x_1) + a_{22}f_{22}(x_2) + \dots\dots\dots + a_{2n}f_{2n}(x_n) &= b_2 \\ \cdot & \\ \cdot & \\ a_{n1}f_{n1}(x_1) + a_{n2}f_{n2}(x_2) + \dots\dots\dots + a_{nn}f_{nn}(x_n) &= b_n \end{aligned} \right\} \dots\dots\dots(1.2)$$

then, the system(1.2) can be called as semi-nonlinear system.

In the system (1.2), the functions  $f_{11}(x_1), f_{22}(x_2), \dots, f_{nn}(x_n)$  are linear in  $x_1, x_2, \dots, x_n$ , then the system (1.2) can be called as semi non-linear system with linear diagonals.

We now write the semi non-linear system with linear diagonals as

$$\left. \begin{aligned} a_{11}x_1 + a_{12}f_{12}(x_2) + \dots\dots\dots + a_{1n}f_{1n}(x_n) &= b_1 \\ a_{21}f_{21}(x_1) + a_{22}x_2 + \dots\dots\dots + a_{2n}f_{2n}(x_n) &= b_2 \\ \cdot & \\ \cdot & \\ a_{n1}f_{n1}(x_1) + a_{n2}f_{n2}(x_2) + \dots\dots\dots + a_{nn}x_n &= b_n \end{aligned} \right\} \dots\dots\dots(1.3)$$

We assume throughout this paper that the matrix obtained from (1.3) i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix} \dots\dots\dots(1.4)$$

is a positive definite matrix.

## II. ITERATIVE METHODS

we now discuss the Jacobi, Gauss-seidel and SOR methods for solving the semi-non linear system with linear diagonals i.e., the system (1.3).

### A. Jacobi Method

Firstly, we re-write system (1.3) as

$$\left. \begin{aligned} x_1 + \frac{a_{12}}{a_{11}} f_{12}(x_2) + \dots\dots\dots + \frac{a_{1n}}{a_{11}} f_{1n}(x_n) &= \frac{b_1}{a_{11}} \\ \frac{a_{21}}{a_{22}} f_{21}(x_1) + x_2 \dots\dots\dots + \frac{a_{2n}}{a_{22}} f_{2n}(x_n) &= \frac{b_2}{a_{22}} \\ \cdot & \\ \cdot & \\ \frac{a_{n1}}{a_{nn}} f_{n1}(x_1) + \frac{a_{n2}}{a_{nn}} f_{n2}(x_2) \dots\dots\dots + x_n &= \frac{b_n}{a_{nn}} \end{aligned} \right\} \dots\dots\dots(2.1)$$

Forming a matrix  $A_s$  by collecting the coefficients of the variables as well as functions, we have

$$A_s = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdot & \cdot & \cdot & \frac{a_{12}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 1 & \cdot & \cdot & \cdot & \frac{a_{2n}}{a_{22}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdot & \cdot & \cdot & 1 \\ a_{nn} & a_{nn} & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \dots\dots\dots (2.2)$$

Splitting the matrix  $A_s$  as

$$A_s = I - L_s - U_s \dots\dots\dots (2.3)$$

where  $-L_s$  and  $-U_s$  are strictly lower and upper triangular parts of the matrix  $A_s$  respectively. Now, the Jacobi

matrix for the semi non-linear system (2.1) is

$$J_s = (L_s + U_s) \dots\dots\dots (2.4) \quad \text{Let } \lambda_i \text{ be the eigenvalues of the jacobi matrix } J_s \text{ such that}$$

$$-1 < \lambda_i < 1 \dots\dots\dots (2.5)$$

Let the maximum eigen values of the matrix  $J_s$  in magnitude i.e., the spectral radius of  $J_s$  be  $\overline{\mu}_s$ . Then ,we have

$$\rho(J_s) = \max |\lambda_i(J_s)| = \overline{\mu}_s \dots\dots\dots (2.6)$$

(i = 1, 2, ..., n)

The Jacobi method for the solution of the system (2.1) is given by

$$\left. \begin{aligned} x_1^{(k+1)} &= (b_1 - a_{12}f_{12}(x_2^k) - \dots\dots\dots - a_{1n}f_{1n}(x_n^k)) / a_{11} \\ x_2^{(k+1)} &= (b_2 - a_{21}f_{21}(x_1^k) - \dots\dots\dots - a_{2n}f_{2n}(x_n^k)) / a_{22} \\ &\cdot \\ x_n^{(k+1)} &= (b_n - a_{n1}f_{n1}(x_1^k) - \dots\dots\dots - a_{n,n-1}f_{n,n-1}(x_{n-1}^k)) / a_{nn} \end{aligned} \right\} \dots\dots\dots (2.7)$$

$$(k = 0, 1, 2, \dots)$$

This method (2.7) converges as long as  $\overline{\mu}_s$  of (2.6) is less than one.

**B. Gauss-Seidel Method**

The Gauss-Seidel method for the system (2.1) is given by

$$\left. \begin{aligned} x_1^{(k+1)} &= (b_1 - a_{12}f_{12}(x_2^k) - a_{13}f_{13}(x_3^k) \dots\dots\dots - a_{1n}f_{1n}(x_n^k)) / a_{11} \\ x_2^{(k+1)} &= (b_2 - a_{21}f_{21}(x_1^{k+1}) - a_{23}f_{23}(x_3^k) \dots\dots\dots - a_{2n}f_{2n}(x_n^k)) / a_{22} \\ &\cdot \\ &\cdot \\ x_n^{(k+1)} &= (b_n - a_{n1}f_{n1}(x_1^{k+1}) - a_{n2}f_{n2}(x_2^{k+1}) \dots\dots\dots - a_{n,n-1}f_{n,n-1}(x_{n-1}^{k+1})) / a_{nn} \end{aligned} \right\} \dots\dots\dots (28)$$

$$(k = 0, 1, 2, \dots)$$

The Gauss-Seidel iterative matrix is

$$G_s = (I - L_s)^{-1} U_s \dots\dots\dots (2.9)$$

where  $L_s$  and  $U_s$  are as defined in (2.3).

This method converges as long as the spectral radius of  $G_s$  in magnitude is less than one i.e.,

$$\rho(G_s) < 1 \dots\dots\dots (2.10)$$

**C. Successive Over Relaxation (SOR) Method**

The SOR method for the solution of (2.1) is given by

$$\left. \begin{aligned} x_1^{(k+1)} &= (1 - \omega)x_1^{(k)} - \omega \frac{a_{12}}{a_{11}} f_{12}x_2^{(k)} - \dots\dots\dots - \omega \frac{a_{1n}}{a_{11}} f_{1n}x_n^k + \omega b_1 \\ x_2^{(k+1)} &= -\omega \frac{a_{21}}{a_{22}} f_{21}x_1^{(k+1)} + (1 - \omega)x_2^{(k)} - \dots\dots\dots - \omega \frac{a_{2n}}{a_{22}} f_{2n}x_n^k + \omega b_2 \\ &\vdots \\ x_n^{(k+1)} &= -\omega \frac{a_{n1}}{a_{nn}} f_{n1}x_1^{(k+1)} - \omega \frac{a_{n2}}{a_{nn}} f_{n2}x_2^{(k+1)} - \dots\dots\dots - \omega \frac{a_{n,n-1}}{a_{nn}} f_{n,n-1}x_{n-1}^{k+1} + (1 - \omega)x_n^k + \omega b_n \end{aligned} \right\} \dots\dots(2.11)$$

(k = 0, 1, 2, \dots\dots\dots)

where, the choice for the relaxation parameter  $\omega$  of SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - (\overline{\mu}_s)^2}} \dots\dots\dots(2.12)$$

where  $\overline{\mu}_s$  is as defined in (2.6).

The SOR method (2.11) in matrix notation is

$$\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ \omega \frac{a_{21}}{a_{22}} \frac{f_{21}(x_1)}{x_1} & 1 & \dots & \dots & 0 \\ \omega \frac{a_{31}}{a_{33}} \frac{f_{31}(x_1)}{x_1} & \omega \frac{a_{32}}{a_{33}} \frac{f_{32}(x_2)}{x_2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega \frac{a_{n1}}{a_{nn}} \frac{f_{n1}(x_1)}{x_1} & \omega \frac{a_{n2}}{a_{nn}} \frac{f_{n2}(x_2)}{x_2} & \dots & \dots & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}^{(k+1)} =$$

$$\begin{bmatrix} 1-\omega & -\omega \frac{a_{12}}{a_{11}} \frac{f_{12}(x_2)}{x_2} & \dots & -\omega \frac{a_{1n}}{a_{11}} \frac{f_{1n}(x_n)}{x_n} \\ 0 & 1-\omega & \dots & -\omega \frac{a_{2n}}{a_{22}} \frac{f_{2n}(x_n)}{x_n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1-\omega \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}^{(k)} + \omega \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

The SOR iterative matrix is

$$S_s = (I - \omega L_s)^{-1} \{ (1 - \omega)I + \omega U_s \} \dots\dots (2.13)$$

This method converges if

$$\rho(S_s) < 1 \dots\dots (2.14)$$

### III. NUMERICAL EXAMPLES

#### A. Example 3.1

We consider a semi non-linear system with linear diagonals i.e.,

$$\left. \begin{aligned} 20x_1 - x_2^3 - x_3^2 &= 18 \\ -x_1^3 + 7x_2 - 2x_3 &= 4 \\ -x_1^2 - 2x_2^2 + 10x_3 &= 7 \end{aligned} \right\} \dots\dots (3.1)$$

whose exact solution is a unit vector.

The matrix  $A_s$  for the system (3.1) as obtained in (2.2) i.e.,

$$A_s = \begin{bmatrix} 20 & -1 & -1 \\ -1 & 7 & -2 \\ -1 & -2 & 10 \end{bmatrix} \dots\dots (3.2)$$

is positive definite and the eigen values of Jacobi matrix  $J_s$  i.e.,

$$J_s = \begin{bmatrix} 0 & 1/20 & 1/20 \\ 1/7 & 0 & 2/7 \\ 1/10 & 2/10 & 0 \end{bmatrix} \dots\dots (3.3)$$

are 0.281822, 0.042332 and 0.239490 and hence  $\overline{\mu}_s = 0.281822$ . The relaxation parameter  $\omega_s$  of SOR method as defined in (2.12),

is obtained as 
$$\omega_s = 1.02068588 \dots\dots (3.4)$$

The methods discussed in this paper are applied to obtain the solution of (3.1) up to an error less than  $0.5 \times 10^{-9}$  taking a null vector as an initial guess and

the results obtained are tabulated below along with the error 
$$E = \sqrt{\sum_{i=1}^n |1 - x_i|}$$
.

Table-1 Iterative compressions

Methods	No. Of iterations took for the convergence (n)	Error (E)
Jacobi	30	0.33675133e <sup>-4</sup>
Gauss-Seidel	17	0.1083196e <sup>-4</sup>
SOR	15	0.15692699e <sup>-4</sup>

**B. Example 3.2**

For the following semi nonlinear system with linear diagonals

$$\left. \begin{aligned} 20x_1 - x_2^3 - x_3^2 &= 18 \\ -x_1^2 - 2x_2 + 10x_3 &= 7 \\ -x_1^3 + 7x_2 - 2x_3 &= 4 \end{aligned} \right\} \dots\dots\dots(3.2)$$

the coefficient matrix  $A_s$  is

$$A_s = \begin{bmatrix} 20 & -1 & -1 \\ -1 & -2 & 10 \\ -1 & 7 & -2 \end{bmatrix} \dots\dots\dots(3.3)$$

and the jacobi matrix  $J_s$  is

$$J_s = \begin{bmatrix} 0 & 0.05 & 0.05 \\ 0.5 & 0 & -5 \\ 0.5 & -3.5 & 0 \end{bmatrix} \dots\dots\dots(3.4)$$

It is calculated that the eigen values of  $A_s$  and  $J_s$  are 20.1458, 6.2203, -10.3661 and 0.012108, -4.195313, 4.183205 respectively. And hence, the matrix  $A_s$  is not positive definite and the eigen values of  $J_s$  are not less than unity in magnitude.

**IV. CONCLUSION**

As seen in the above tabulated results that the Jacobi, Gauss-Seidel and SOR methods works well as long as the matrix A of (1.4) is positive definite and it is also observed from example(3.2) that all the methods discussed in this paper diverged if A of(1.4) is not positive definite.

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