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Fixed Point Results in Quasimetric Spaces

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Abstract: In the setting of quasimetric spaces, we prove some new results on the existence of fixed points for contractive type maps with respect to Q -function. Our results either improve or generalize many known results in the literature.

Keywords: Quasimetric space, contractive maps, fixed point, Q -function.

I. INTRODUCTION AND PRELIMINARIES

Let X be a metric space with metric d . We use $S(X)$ to denote the collection of all nonempty subsets of X , $cl(X)$ for the collection of all nonempty closed subsets of X , $CB(X)$ for the collection of all nonempty closed bounded subsets of X and H for the Hausdorff metric on $CB(X)$ that is,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, A, B \in CB(X),$$

Where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B .

For a multivalued map $T : X \rightarrow CB(X)$, we say

(a) T is contraction [1] if there exists a constant $\lambda \in (0, 1)$, such that for all $x, y \in X$,
 $H(T(x), T(y)) \leq \lambda d(x, y)$,

(b) T is weakly contractive [2] if there exist constants $h, b \in (0, 1)$, $h < b$, such that for any $x \in X$, there is $y \in I_b^x$ satisfying
 $d(y, T(y)) \leq hd(x, y)$,

where $I_b^x = \{y \in T(x) : bd(x, y) \leq d(x, T(x))\}$.

A point $x \in X$ is called a fixed point of a multivalued map $T : X \rightarrow S(X)$ if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$.

A sequence $\{x_n\}$ in X is called an orbit of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all integer $n \geq 1$. A real valued function f on X is called lower semi continuous if for any sequence $\{x_n\} \subset X$ with $x_n \rightarrow x \in X$ implies that $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

Using the Hausdorff metric, Nadler Jr. [1] has established a multivalued version of the well-known Banach contraction principle in the setting of metric spaces as follows.

A. Theorem

Let (X, d) be a complete metric space, then each contraction map $T : X \rightarrow CB(X)$ has a fixed point.

Without using the Hausdorff metric, Feng and Liu [2] generalized Nadler's contraction principle as follows.

B. Theorem

Let (X, d) be a complete metric space and let $T : X \rightarrow cl(X)$ be a weakly contractive map, then T has a fixed point in X provided the real valued function $f(x) = d(x, T(x))$ on X is a lower semicontinuous.

In [3], Kada et al. introduced the concept of ω -distance in the setting of metric spaces as follows.

A function $\omega : X \times X \rightarrow [0, \infty)$ is called a ω -distance on X if it satisfies the following:

(w1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$, for all $x, y, z \in X$

(w2) ω is lower semicontinuous in its second variable;

(w3) for any $\epsilon > 0$, there exists $\delta > 0$, such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Note that in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$, and not either of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily

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holds. Clearly, the metric d is a ω -distance on X . Many other examples and properties of ω -distances are given in [3]. In [4], Suzuki and Takahashi improved Nadler contraction principle (Theorem 1.1) as follows.

C. Theorem

Let (X, d) be a complete metric space and let $T : X \rightarrow cl(X)$. If there exist a ω -distance ω on X and a constant $\lambda \in (0, 1)$, such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$\omega(u, v) \leq \lambda \omega(x, y),$$

then T has a fixed point.

Recently, Latif and Albar [5] generalized Theorem 1.2 with respect to ω -distance (see, Theorem 3.3 in [5]), and Latif [6] proved a fixed point result with respect to ω -distance (see, Theorem 2.2 in [6]) which contains Theorem 1.3 as a special case.

A nonempty set X together with a quasimetric d (i.e., not necessarily symmetric) is called a quasimetric space. In the setting of a quasimetric spaces, Al-Homidan et al. [7] introduced the concept of a q -function on quasimetric spaces which generalizes the notion of a ω -distance.

A function $q : X \times X \rightarrow [0, \infty)$ is called a q -function on X if it satisfies the following conditions:

(Q1) $q(x, z) \leq q(x, y) + q(y, z)$, for all $x, y, z \in X$,

(Q2) If $\{y_n\}$ is a sequence in X such that $y_n \rightarrow y \in X$ and for $x \in X$, $q(x, y_n) \leq M$ for some $M = M(x) > 0$, then $q(x, y) \leq M$,

(Q3) for any $\epsilon > 0$, there exists $\delta > 0$, such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \epsilon$.

Note that every ω -distance is a q -function, but the converse is not true in general [7]. Now, we state some useful properties of q -function as given in [7].

D. Lemma

Let (X, d) be a complete quasimetric space and let q be a Q -function on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, \infty)$ converging to 0, then the following hold for any $x, y, z \in X$.

if $q(x_n, y) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $n \geq 1$, then $y = z$ in particular, if $q(x, y) = 0$ and $q(x, z) = 0$,

then $y = z$, if $q(x_n, y_n) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $n \geq 1$, then $\{y_n\}$ converges to Z ;

if $q(x_n, x_m) \leq \alpha_n$ for any $n, m \geq 1$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

if $q(y, x_n) \leq \alpha_n$ for any $n \geq 1$, then $\{x_n\}$ is a Cauchy sequence.

Using the concept Q -function, Al-Homidan et al. [7] recently studied an equilibrium version of the Ekeland-type variational principle. They also generalized Nadler's fixed point theorem (Theorem 1.1) in the setting of quasimetric spaces as follows.

E. Theorem

Let (X, d) be a complete quasimetric space and let $T : X \rightarrow cl(X)$. If there exist Q -function q on X and a constant $\lambda \in (0, 1)$, such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$q(u, v) \leq \lambda q(x, y),$$

then T has a fixed point.

In the sequel, we consider X as a quasimetric space with quasimetric d .

Considering a multivalued map $T : X \rightarrow S(X)$, we say

T is weakly q -contractive if there exist Q -function q on X and constants $h, b \in (0, 1)$, $h < b$, such that for any $x \in X$, there is $y \in J_b^x$ satisfying

$$q(y, T(y)) \leq hq(x, y),$$

where $J_b^x = \{y \in T(x) : bq(x, y) \leq q(x, T(x))\}$ and $q(x, T(x)) = \inf \{q(x, y) : y \in T(x)\}$,

T is generalized q -contractive if there exists a Q -function q on X , such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

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$$q(u, v) \leq kq(x, y),$$

where k is a function of $[0, \infty)$ to $[0, 1)$, such that $\limsup_{r \rightarrow t^+} k(r) < 1$ for all $t \geq 0$.

Clearly, the class of *weakly* q -contractive maps contains the class of weakly contractive maps, and the class of generalized q -contractive maps contains the classes of generalized ω -contraction maps [6], ω -contractive maps [4], and q -contractive maps [7].

In this paper, we prove some new fixed point results in the setting of quasimetric spaces for weakly q -contractive and generalized q -contractive multivalued maps. Consequently, our results either improve or generalize many known results including the above stated fixed point results.

II. THE MAIN RESULTS

First, we prove a fixed point theorem for weakly q -contractive maps in the setting of quasimetric spaces.

A. Theorem

Let X be a complete quasimetric space and let $T : X \rightarrow cl(X)$ be a weakly q -contractive map. If a real valued function $f(x) = q(x, T(x))$ on X is lower semicontinuous, then there exists $y_0 \in X$, such that $q(y_0, T(y_0)) = 0$. Further, if $q(x_0, y_0) = 0$, then y_0 is a fixed point of T .

B. Proof

Let $x_0 \in X$, Since T is weakly contractive, there is $x_1 \in J_b^{x_0} \subseteq T(x_0)$, such that

$$q(x_1, T(x_1)) \leq hq(x_0, x_1),$$

where $h < b$. Continuing this process, we can get an orbit $\{x_n\}$ of T at x_0 satisfying $x_{n+1} \in J_b^{x_n}$ and

$$q(x_{n+1}, T(x_{n+1})) \leq hq(x_n, x_{n+1}), n = 0, 1, 2, \dots$$

Since $bq(x_n, x_{n+1}) \leq q(x_n, T(x_n))$, and $h < b < 1$, thus we get

$$q(x_{n+1}, T(x_{n+1})) \leq q(x_n, T(x_n)),$$

If we put $a = \frac{h}{b}$, then also we have

$$q(x_{n+1}, T(x_{n+1})) \leq aq(x_n, T(x_n)),$$

Thus, we obtain

$$q(x_n, T(x_n)) \leq a^n q(x_0, T(x_0)), n = 0, 1, 2, \dots,$$

and since $0 < a < 1$, hence the sequence $\{f(x_n)\} = q(x_n, T(x_n))$, which is decreasing, converges to 0. Now, we show that $\{x_n\}$ is a Cauchy sequence. Note that

$$q(x_n, x_{n+1}) \leq a^n q(x_0, x_1), n = 0, 1, 2, \dots,$$

Now, for any integer $n, m \geq 1$ with $m > n$, we have

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \dots + q(x_{m-1}, x_m) \\ &\leq a^n q(x_0, x_1) + a^{n+1} q(x_0, x_1) + \dots + a^{m-1} q(x_0, x_1) \\ &\leq \frac{a^n}{1-a} q(x_0, x_1), \end{aligned}$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X , there exists some $y_0 \in X$, such that $\lim_{n \rightarrow \infty} x_n = y_0$. Now, since f is lower semicontinuous, we have

$$0 \leq f(y_0) \leq \lim_{n \rightarrow \infty} \inf f(x_n) = 0,$$

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and thus, $f(y_0) = q(y_0, T(y_0)) = 0$. It follows that there exists a sequence $\{y_n\}$ in $T(y_0)$, such that $q(y_0, y_n) \rightarrow 0$. Now, if $q(y_0, y_n) = 0$, then by Lemma 1.4, $y_n \rightarrow y_0$. Since $T(y_0)$ is closed, we get $y_0 \in T(y_0)$.

Now, we prove the following useful lemma.

C. Lemma

Let (X, d) be a complete quasimetric space and let $T : X \rightarrow cl(X)$ be a generalized q -contractive map, then there exists an orbit $\{x_n\}$ of T at x_0 , such that the sequence of nonnegative numbers $\{q(x_n, x_{n+1})\}$ is decreasing to zero and $\{x_n\}$ is a Cauchy sequence.

D. Proof

Let x_0 be an arbitrary but fixed element of X and let $x_1 \in T(x_0)$. Since T is generalized as a q -contractive, there is $x_2 \in T(x_1)$, such that

$$q(x_1, x_2) \leq kq(x_0, x_1),$$

Continuing this process, we get a sequence $\{x_n\}$ in X , such that $x_{n+1} \in T(x_n)$ and

$$q(x_n, x_{n+1}) \leq kq(x_{n-1}, x_n),$$

Thus, for all $n \geq 1$, we have

$$q(x_n, x_{n+1}) < q(x_{n-1}, x_n),$$

Write $t_n = q(x_n, x_{n+1})$. Suppose that $\lim_{n \rightarrow \infty} t_n = \lambda > 0$, then we have

$$t_n \leq k t_{n-1}$$

Now, taking limits as $n \rightarrow \infty$ on both sides, we get

$$\lambda \leq \lim_{n \rightarrow \infty} \sup k(t_{n-1})\lambda < \lambda,$$

which is not possible, and hence the sequence of nonnegative numbers $\{t_n\}$ which is decreasing, converges to 0. Finally, we show that $\{x_n\}$ is a Cauchy sequence. Let $\alpha = \lim_{r \rightarrow 0^+} \sup k(r) < 1$. There exists real number β such that $\alpha < \beta < 1$. Then for sufficiently large n , $k(t_n) < \beta$, and thus for sufficiently large n , we have $t_n < \beta t_{n-1}$. Consequently, we obtain $t_n < \beta^n t_0$, that is,

$$q(x_n, x_{n+1}) < \beta^n q(x_0, x_1), n = 0, 1, 2, \dots,$$

Now, for any integers $n, m \geq 1, m > n$,

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \dots + q(x_{m-1}, x_m) \\ &\leq \beta^n q(x_0, x_1) + \beta^{n+1} q(x_0, x_1) + \dots + \beta^{m-1} q(x_0, x_1) \\ &\leq \frac{\beta^n}{1 - \beta} q(x_0, x_1), \end{aligned}$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence.

Applying Lemma 2.2, we prove a fixed point result for generalized q -contractive maps.

E. Theorem

Let (X, d) be a complete quasimetric space then each generalized q -contractive map $T : X \rightarrow cl(X)$ has a fixed point.

F. Proof

It follows from Lemma 2.2 that there exists a Cauchy sequence $\{x_n\}$ in X such that the decreasing sequence $\{q(x_n, x_{n+1})\}$ converges to 0. Due to the completeness of X , there exists some $y_0 \in X$ such

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that $\lim_{n \rightarrow \infty} x_n = y_0$. Let n be arbitrary fixed positive integer then for all positive integers m with $m > n$, we have

$$q(x_n, x_m) \leq \frac{\beta^n}{1 - \beta} q(x_0, x_1),$$

Let $M = \frac{\beta^n}{1 - \beta} q(x_0, x_1)$, then $M \geq 0$. Now, note that

$$q(x_n, x_m) \leq M \Rightarrow q(x_n, y_0) \leq M,$$

Since n was arbitrary fixed, we have

$$q(x_n, y_0) \leq \frac{\beta^n}{1 - \beta} q(x_0, x_1), \text{ for all positive integer } n,$$

Note that $q(x_n, y_0)$ converges to 0. Now, since $x_n \in T(x_{n-1})$ and T is a generalized q -contractive map, then there is $u_n \in T(y_0)$, such that

$$q(x_n, u_n) \leq kq(x_{n-1}, y_0),$$

And for large n , we obtain

$$q(x_n, u_n) \leq kq(x_{n-1}, y_0) < \beta q(x_{n-1}, y_0),$$

thus, we get

$$q(x_n, u_n) < \beta q(x_{n-1}, y_0) \leq \frac{\beta^n}{1 - \beta} q(x_0, x_1),$$

Thus, it follows from Lemma 1.4 that $u_n \rightarrow y_0$. Since $T(y_0)$ is closed, we get $y_0 \in T(y_0)$.

Corollary 2.4.

Let (X, d) be a complete quasimetric space and q a Q -function on X . Let $T : X \rightarrow cl(X)$ be a multivalued map, such that for any $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ with

$$q(u, v) \leq kq(x, y),$$

Where k is a monotonic increasing function from $(0, \infty)$ to $[0, 1)$, then T has a fixed point.

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