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Fixed Point Theorems on Multi Valued Mappings in B-Metric Spaces

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Abstract: In this paper, we prove a fixed point theorem and a common fixed point theorem for multi valued mappings in complete b-metric spaces.
Keywords: b-Metric space, Multi-valued mappings, Contraction, Fixed point

I. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays one of the important roles in nonlinear analysis. It has been applied in physical sciences, computing sciences and Engineering. In 1922, Stefan Banach proved a famous fixed point theorem for contractive mappings in complete metric spaces. Later, Czerwik (1993, 1998) has come up with b-metrics which generalized usual metric spaces. After his contribution, many results were presented in β-generalized weak contractive multifunctions and b-metric spaces (Alikhani et al. 2013; Boriceanu 2009; Mehemet and Kiziltunc 2013). The following definitions will be needed in the sequel:

A. Definition

Nadler (1969) Let X and Y be nonempty sets. T is said to be multi-valued mapping from X to Y if T is a function for X to the power set of Y. we denote a multi-valued map by:

$$T{:}X \to 2^Y.$$

B. Definition

Nadler (1969) A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mapping T if $x_0 \in Tx_0$.

C . Example

Joseph (2013) Every single valued mapping can be viewed as a multi-valued mapping. Let $f{:}X \to Y$ be a single valued mapping. Define $T{:}X \to 2^Y$ by $Tx = \{f(x)\}$. Note that T is a multi-valued mapping iff for each $x \in X$, $TX \subseteq Y$. Unless otherwise stated we always assume Tx is non-empty for each $x, y \in X$.

D. Definition

Banach (1922) Led (X, d) be a metric space. A map $T{:}X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda d(x, y)$, for all $x, y \in X$.

E. Definition

Nadler (1969) Let (X, d) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by d. That is

$$H(A, B) = max\{\sup x{\in}A\ d(x, B), \sup y{\in}B\ d(y, A)\}$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X and $d(x, B) = \inf\{d(x, b){:}\ b \in B\}$, for all $x \in X$.

E. Definition

Nadler (1969) Let (X, d) be a metric space. A map $T{:}X \to CB(X)$ is said to be multi valued contraction if there exists $0 \le \lambda < 1$ such that $H(Tx, Ty) \le \lambda d(x, y)$, for all $x, y \in X$

F. Lemma

Nadler (1969) *If $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \le H(A, B) + \epsilon$.*

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G. Definition

Aydi et al. (2012) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d:X \times X \to \mathbb{R}^+$ is called a b-metric provide that, for all $x, y, z \in X$,

$d(x, y) = 0$ if and only if $x = y$

$d(x, y) = d(y, x)$

$d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair(X, d) is called a b-metric space.

H. Example

Boriceanu (2009) The space $l_p(0 < p < 1)$, $lp=\{(xn: \sum_{n=1}^{\infty}|x_n|^p < \infty\}$, together with the function $d:l_p \times l_p \to \mathbb{R}^+$.

I. Example

Boriceanu (2009) The space $L_p(0 < p < 1)$ for all real function $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take $d(x, y) = (\int_0^1 |x(t) - y(t)| dt)^{\frac{1}{p}}$.

J .Example

Aydi et al. (2012) Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$, $d(0, 1) = d(1, 2) = d(0, 1) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then $d(x, y) \leq \frac{m}{2}[d(x, z)+d(z, y)]$ for all $x, y, z \in X$. If $m > 2$, the ordinary triangle inequality does not hold.

K. Definition

Let (X, d) be a b-metric space. Then a sequence (x_n) in X is called Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $m, n \geq n(\epsilon)$ we have $d(x_n, x_m) < \epsilon$.

L. Definition

Let be a (X, d) b-metric space. Then a sequence (x_n) in X is called convergent sequence if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$ we have $d(x_n, x) < \epsilon$. In this case we write $\lim n \to \infty\, xn = x$

Our first result is the following theorem.

II. MAIN RESULTS

A. Definition

Let (X, d) be a b-metric space with constant $s \geq 1$. A map $T:X \to CB(X)$ is said to be multi valued generalized contraction if

$$H(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y) + a_6 \frac{d(x,Tx)\,(1+d(x,Tx))}{1+d(x,y)} ,(1)$$

for all $x, y \in X$ and $a_i \geq 0$, $i = 1, 2, 3, \ldots 6$ with $a_1 + a_2 + 2sa_3 + a_4 + a_5 + a_6 < 1$.

B. Theorem

Let (X, d) be a complete b-metric space with constant$s \geq 1$. Let $T:X \to CB(X)$ be a multi valued generalized contraction mapping. Then T has a unique fixed point.

C. Proof

Fix any $x \in X$. Define $x_0 = x$ and let $x_1 \in Tx_0$. By Lemma 7, we may choose $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq H(Tx_0, Tx_1) + (a_1 + sa_3 + a_5 + a_6)$.

Now,

$$d(x_1, x_2) \leq H(Tx0, Tx1)+(a_1+sa_3+a_5+a_6)$$

$$\leq a_1d(x_0, Tx_0)+a_2d(x_1, Tx_1)+a_3d(x_0, Tx_1)+a_4d(x_1, Tx_0)+a_5d(x_0, x_1)+a_6 \frac{d(x,Tx)\,(1+d(x,Tx))}{1+d(x,y)} +(a_1+sa_3+a_5+a_6)$$

$$d(x_1, x_2) \leq a_1d(x_0, x_1)+ a_2d(x_1, x_2)+ a_3d(x_0, x_2)+ a_4d(x_1, x_1)+ a_5d(x_0, x_1)+ a_6d(x_0, x_1)+ (a_1+sa_3+a_5+a_6)$$

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$\leq (a_1+a_5+a_6)\, d(x_0, x_1)+a_2d(x_1, x_2)+a_3s[d(x_0, x_1)+d(x_1, x_2)]+(a_1+sa_3+a_5+a_6)$

$\leq(a_1+sa_3+a_5+a_6)d(x_0, x_1)+a2d(x_1, x_2)+sa_3d(x_1, x_2)+(a_1+sa_3+a_5+a_6)$

$d(x_1, x_2) \leq \frac{(a1+sa3+a5+a6)}{1-(a2+Sa3)}\, d(x_0, x_1) + \frac{(a1+sa3+a5+a6)}{1-(a2+Sa3)}$

By Lemma 7, there exist $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq d(Tx_1, x_2) + \frac{(a1+sa3+a5+a6)^2}{1-(a2+Sa3)}$.

Now,

$$d(x_2, x_3)\leq H(Tx_1, x_2)+ \frac{(a1+sa3+a5+a6)^2}{1-(a2+Sa3)}\quad a_1d(x_1, Tx_1)+a_2d(x_1, Tx_2)+ a_3d(x_1, Tx_2) + a_4d(x_2, Tx_1) + a_5d(x_1, x_2) + a_6d(x_1, x_2) +$$

$$\frac{(a1+sa3+a5+a6)}{1-(a2+Sa3)}\, d(x_1, x_2) + \frac{(a1+sa3+a5+a6)^2}{(1-(a2+Sa3))^2}$$

$d(x_2, x_3) \leq \left(\frac{(a1+sa3+a5+a6)}{1-(a2+Sa3)}\right)^2 d(x_0, x_1) + 2\left[\frac{(a1+sa3+a5+a6)}{1-(a2+Sa3)}\right]^2$

Continuing this process, we obtain by induction a sequence $\{x_n\}$ such that $x_n \in Tx_{n-1}$, $x_{n+1} \in Tx_n$ such that

$$d(xn, xn+1)\leq \frac{(a1+sa3+a5+a6)}{1-(a2+sa3)}d(xn-1,xn)+ \left[\frac{(a1+sa3+a5+a6)}{(1-(a2+sa3)}\right]^n$$

for all $n \in \mathbb{N}$ and let $k = \frac{(a1+sa3+a5+a6)}{1-(a2+sa3)}$

$$d(xn, xn+1) \leq kd(x_{n-1}, x_n)+k^n$$
$$\leq k[kd(x_{n-2}, x_{n-1})+ k^{n-1}]+ k^n$$
$$= k^2d(x_{n-2}, x_{n-1})+kk^{n-1}+k^n$$

.

$$d(x_n, x_{n+1}) \leq k^nd(x_0, x_1)+nk^n$$

Since $k < 1$, $\sum k^n$ and $\sum nk^n$ have same radius of convergence. Then, $\{x_n\}$ is a Cauchy sequence. But (X, d) is a complete b-metric space, it follows that $\{xn\}\infty n=0$ is convergent.

$$u = \lim n\to\infty xn.$$

Now,

$$d(u, Tu) \leq s[d(u, xn+1)+d(xn+1, Tu)]$$
$$d(u, Tu) \leq s[d(u, xn+1)+d(Txn, Tu)]$$

Using (1), we obtain,

$$d(u,Tu)\leq s[d(u,xn+1)]+s[a1d(xn,Txn)+a2d(u,Tu)+a3d(xn,Tu)+a4d(u,Txn)+a5d(xn,u)+a6d(xn,u)].$$

As $n\to\infty$,

$$d(u, Tu) \leq s[a2d(u, Tu)+a3d(u, Tu)](1-(a2s+a3s))d(u, Tu)\leq 0.$$

The above inequality is true unless $d(u, Tu) = 0$. Thus, $Tu = u$.

Now we show that u is the unique fixed point of T. Assume that v is another fixed point of T. Then we have $Tv = v$ and

$$d(u, v) = d(Tu, Tv)\leq s[d(u, Tv)+d(v, Tu)]$$

we obtain, $d(u, v) \leq 2sd(u, v)$. This implies that $u = v$. This completes the proof.

D. Theorem

Let (X, d) *be a complete b-metric space with constant* $\lambda \geq 1$. *Let* $T, S{:}X \to CB(X)$ *be a multi valued mapping satisfies the condition*:

$$H(Tx, Sy) \leq a_1d(x, Tx)+a_2d(y, Sy)+a_3d(x, Sy)+a_4d(y, Tx)+a_5d(x, y),$$

for all $x,y \in X$ *and* $a_i \geq 0$, $i = 1, 2, \ldots 5$, *with* $(a_1 + a_2)(\lambda + 1) + (a_3 + a_4)(\lambda^2 + \lambda) + 2\lambda a_5 < 2$, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.*Then T and S have a unique common fixed point.*

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E. Proof

Fix any $x \in X$. Define $x_0 = x$ and let $x_1 \in Tx_0$, $x_2 \in Sx$ such that $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, By Lemma 7, we may choose $x_2 \in Sx_1$ such that $d(x_1, x_2) \le H(Tx_0, Sx_1) + (a_1 + a_5 + \lambda a_3)$

$d(x_1, x_2) \le a_1d(x_0, Tx_0) + a_2d(x_1, Sx_1) + a_3d(x_0, Sx_1) + a_4d(x_1, Tx_0) + a_5d(x_0, x_1) + (a_1+a_5+\lambda a_3)$

$= a_1d(x_0, x_1)+a_2d(x_1, x_2)+a_3d(x_0, x_2)+a_4d(x_0, x_1)+a_5d(x_0, x_1)+(a_1+a_5+\lambda a_3)$

$\le a_1d(x_0, x_1)+a_2d(x_1, x_2)+a_3\lambda[d(x_0, x_1)+d(x_1, x_2)]+a_5d(x_0, x_1)+(a_1+a_5+\lambda a_3)$ (2)

$d(x_1, x_2) \le (a_1+\lambda a_3+a_5)d(x_0, x_1)+(a_2+\lambda a_3)d(x_1, x_2)+(a_1+a_5+\lambda a_3)$

$\le \frac{(a1+a5+\lambda a3)}{1-(a2+\lambda a3)} d(x_0,x_1) + \frac{(a1+a5+\lambda a3)}{1-(a2+\lambda a3)}$

On the other hand and by symmetry, we have

$d(x_2, x_1) = d(Sx_1, Tx_0)$

$\le H(Sx_1, Tx_0)+(a_2+a_5+\lambda a_4) \quad \le a_1d(x_1,Sx_1)+a_2d(x_0,Tx_0)+a_3d(x_1,Tx_0)+a_4d(x_0,Sx_1)+a_5d(x_1,x_0)+(a_2+a_5+\lambda a_4)$

$=a_1d(x_1,x_2)+a_2d(x_0,x_1)+a_3d(x_1,x_1)+a_4d(x_0,x_2)+a_5d(x_0,x_1)+(a_2+a_5+\lambda a_4)$ (3)

$\le a_1d(x_1,x_2)+a_2d(x_0,x_1)+a_4[d(x_0,x_1)+d(x_1,x_2)]+a_5d(x_0,x_1)+(a_2+a_5+\lambda a_4) =(a_2+a_5+\lambda a_4)d(x_0,x_1)+(a_1+\lambda a_4)d(x_2,x_1)(a_2+a_5+\lambda a_4)$

$d(x_2, x_1) \le \frac{(a2+a5+\lambda a4)}{1-(a1+\lambda a4)} d(x_0,x_1) + \frac{(a2+a5+\lambda a4)}{1-(a1+\lambda a4)}$

Adding inequalities (2) and (3) , we obtain

$d(x1,x2) \le \frac{a1+a2+Sa3+Sa4+2a5}{2-(a1+a2+Sa3+Sa4)}d(x_0,x_1) + \frac{(a1+a2+Sa3+Sa4+2a5)}{2-(a1+a2+Sa3+Sa4)}$

Where k = $\frac{(a1+a2+\lambda a3+\lambda a4+2a5}{2-(a1+a2+\lambda a3+\lambda a4)} < \frac{1}{\lambda}$.

Similarly, it can be shown that, there exists $x_3 \in Tx_2$ such that

$d(x_3, x_2) \le H(Tx_2,Sx_1) + \left(\frac{a1+a2+\lambda a3+\lambda a4+2a5}{2-(a1+a2+\lambda a3+\lambda a4)} \right)^2$

$\le k^2 d(x_1,x_0)+2k^2$

Continuing this process, we obtain by induction a sequence $\{x_n\}$ such that $x_{2n+1} \in Tx_{2n}$, $x_{2n+2} \in Sx_{2n+1}$ such that

$d(x_{2n+1}, x_{2n+2}) \le d(Tx_{2n}, Sx_{2n+1}) + (a_1+a_5+\lambda a_3)^{2n+1}$

$\le a_1d(x_{2n}, Tx_{2n})+a_2d(x_{2n+1}, Sx_{2n+1})+a_3d(x_{2n}, Sx_{2n+1})+a_4d(x_{2n+1}, Tx_{2n}) + \quad a_5d(x_{2n}, \; x_{2n+1})+ (a_1+a_5+\lambda a_3)^{2n+1}$ (4)

$d(x_{2n+1}, x_{2n+2}) \le \frac{(a1+a5+\lambda a3)}{1-(a2+\lambda a3)}d(x_{2n},x_{2n+2}) + \frac{(a1+a5+\lambda a3)2n+1}{(1-(a2\lambda a3))2n+1}$

Also,

$$d(x_{2n+2}, x_{2n+1}) \le \frac{(a2+a5+\lambda a4)}{1-(a1+\lambda a4)} d(x_{2n+1},x_{2n}) + \frac{(a2+a5+\lambda a4)^{2n+1}}{(1-(a2\lambda a3))^{2n+1}} \quad (5)$$

From (4) and (5)

$d(x_{2n+1}, x_{2n+2}) \le kd(x_{2n+1}, x_{2n}) + k^{2n+1}$

Therefore,

$d(x_n,x_{n+1}) \le \frac{a1+a2+\lambda a3+\lambda a4+2a5}{2-(a1+a2+\lambda a3+\lambda a4)} d(x_{n-1},x_n) + \frac{a1+a2+\lambda a3+\lambda a4+2a5}{2-(a1+a2+\lambda a3+\lambda a4))n}$ $n \in \mathbf{N}$ and let k = $\frac{a_1+a_2+\lambda a_3+\lambda a_4+2a_5}{2-(a_1+a_2+\lambda a_3+\lambda a_4)}$

$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n)+k^n$

$\le k(d(x^{n-2}, x_{n-1})+k^{n-1})+ k^n$

$=k^2d(x^{n-2}, x^{n-1})+2k^n$

$\le \cdots\cdots\cdots$

$\le k^nd(x_0, x_1)+nk^n$.

Since $0 < k < 1$, $\sum k^n$ and $\sum nk^n$ have same radius of convergence. Then, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \rightarrow z$.

We shall prove that z is a common fixed point of T and S.

$$d(z, Sz) \le \lambda[d(z, x_{2n+1})+d(x_{2n+1}, Tz)]$$
$$\le \lambda[d(z, x_{2n+1})+H(x_{2n+1}, Tz)]$$
$$d(z, Sz) \le \lambda[d(z, x_{2n+1})+d(x_{2n+1}, Sz)]$$
$$\le \lambda[d(z, x_{2n+1})+H(x_{2n}, Sz)] \quad (6)$$

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$$H(x_{2n}, Sz) \leq a_1 d(x_{2n}, Tx_{2n}) + a_2 d(z, Sz) + a_3 d(x_{2n}, Sz) + a_4 d(z, Tx_{2n}) + a_5 d(x_{2n}, z) \quad (7)$$

Using (7) in (6) and letting as $n \to \infty$, we obtain,

$d(z, Sz) \leq \lambda d(z, z) + \lambda[a_1 d(z, z) + a2d(z,Sz) + a_3 d(z, Sz) + a_4 d(z, z) + a_5 d(z, z)]$

$= \lambda[a_2 d(z, Sz) + a_3 d(z, Sz)]$

$\leq \lambda(a_2 + a_3) d(z, Sz)$

$[1 - \lambda(a_2 + a_3)] d(z, Sz) \leq 0.$

$1 - \lambda(a_2 + a_3) \leq 0$ and $S(z)$is closed. Thus, $S(z) = z$.

Similarly, $T(z) = z$.

We show that z is the unique fixed point of S and T. Now,

$$d(z, v) \leq H(Tz, Sv)$$
$$\leq a_1 d(z, Tz) + a\ d(v, Sv) + a_3 d(z, Sv) + a_4 d(v, Tz) + a_5 d(z, v)$$
$$\leq a_3 d(z, v) + a_4 d(z, v) + a_5 d(z, v).$$

Since $[1 - (a_3 + a_4 + a_5)] > 0$, $d(z, v) = 0$. Hence, S and T have a unique common fixed point.

III. CONCLUSION

Many authors have contributed some fixed point results for a self mappings in b-metric spaces. In this paper, we have proved the existence and uniqueness of fixed point results for a multivalued mappings in b-metric spaces. Our contraction mappings also generalize various known contractions like Hardy Roger contraction in the current literature.

REFERENCES

[1] Alikhani H, Gopal D, Miandaragh MA, Rezapour Sh, Shahzad N (2013) Some endpoint results for ββ-generalized weak contractive multifunctions. Sci World J 2013:7. Article ID 948472

[2] Aydi H, et al. A fixed point theorem for set-valued quasi-contractions in b-metric spaces. Fixed Point Theory Appl. 2012;2012:88. doi: 10.1186/1687-1812-2012-88.

[3] Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fundam Math. 1922;3:133–181.

[4] Boriceanu M. Fixed point theory for multivalued generalized contraction on a set with two b-metrics. Stud Univ Babes-Bolyai Math. 2009;LIV(3):1–14.

[5] Czerwik S. Contraction mappings in b-metric spaces. Acta Math Inform Univ Ostraviensis. 1993;1:5–11.

[6] Czerwik S. Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin Math Fis Univ Modena. 1998;46(2):263–276.

[7] Maria Joseph J, Ramganesh E. Fixed point theorem on multi-valued mappings. Int J Anal Appl. 2013;1(2):127–132.

[8] Mehemet K, Kiziltunc H. On some well known fixed point theorems in b-metrics spaces. Turk J Anal Appl. 2013;1(1):13–16.

[9] Nadler SB. Multi-valued contraction mappings. Pac J Math. 1969;30:475–488. doi: 10.2140/pjm.1969.30.475.

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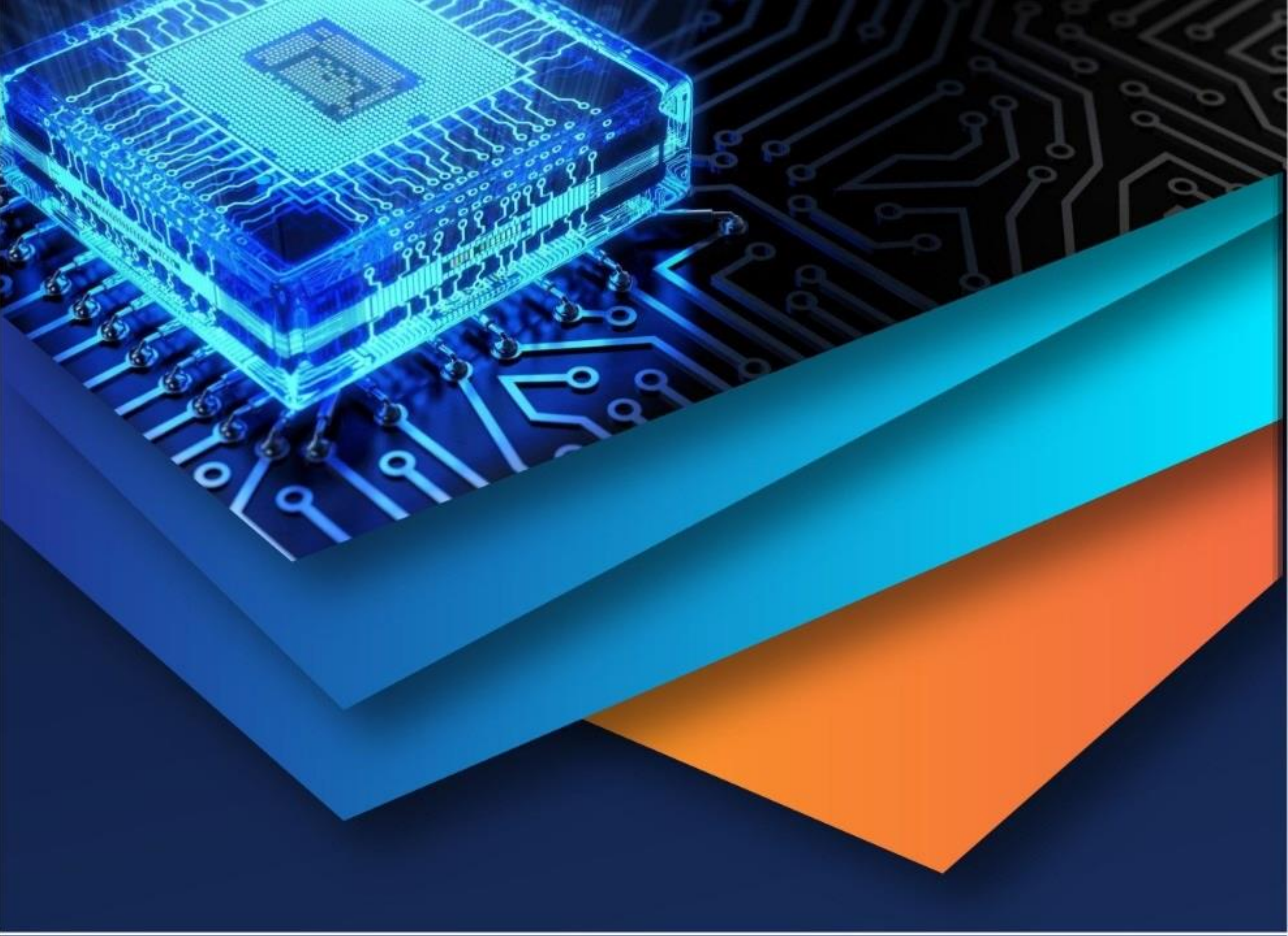

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