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Wardowski Type Fixed Point Theorems in Complete Metric Spaces

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Abstract: In this paper, we state and prove Wardowski type fixed point theorems in metric space by using a modified generalized F-contraction maps. These theorems extend other well-known fundamental metrical fixed point theorems in the literature (Dung and Hang in Vietnam J. Math. 43:743-753, 2015 and Piri and Kumam in Fixed Point Theory Appl. 2014:210, 2014, etc.).

Keywords: fixed point, metric space, F-contraction.

I. INTRODUCTION AND PRELIMINARIES

One of the most well-known results in generalizations of the Banach contraction principle is the Wardowski fixed point theorem [3]. Before providing the Wardowski fixed point theorem, we recall that a self-map T on a metric space (X,d) is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in (0, \infty)$ such that

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))],$$

where \mathcal{F} is the family of all functions $F: (0, \infty) \rightarrow \mathbb{R}$ such that

(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \rightarrow \infty} x_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(x_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Obviously every F-contraction is necessarily continuous. The Wardowski fixed point theorem is given by the following theorem.

A. Theorem 1.1[3]

Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ converges to x^* .

Later, Wardowski and Van Dung [4] have introduced the notion of an F-weak contraction and prove a fixed point theorem for F-weak contractions, which generalizes some results known from the literature. They introduced the concept of an F-weak contraction as follows.

B. Definition 1.2

Let ((X, d) be a metric space. A mapping $T: X \rightarrow X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where

$$I) \quad M(x, y) = \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \}.$$

By using the notion of F-weak contraction, Wardowski and Van Dung [4] have proved a fixed point theorem which generalizes the result of Wardowski as follows.

C. Theorem 1.3[4]

Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ converges to x^* .

Recently, by adding values $d(T^2x, x)$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to (2), Dung and Hang [1] introduced the notion of a modified generalized F -contraction and proved a fixed point theorem for such maps. They generalized an F -weak contraction to a generalized F -contraction as follows.

D. Definition 1.4

Let (X, d) be a metric space. A mapping $T:X \rightarrow X$ is said to be a generalized F -contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(N(x, y))],$$

Where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}.$$

By using the notion of a generalized F -contraction, Dung and Hang have proved the following fixed point theorem, which generalizes the result of Wardowski and Van Dung [4].

E. Theorem 1.5[1]

Let (X, d) be a complete metric space and let $T:X \rightarrow X$ be a generalized F -contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ converges to x^* .

Very recently, Piri and Kumam [2] described a large class of functions by replacing the condition (F_3) in the definition of F -contraction introduced by Wardowski with the following one:

(F_3') :

F is continuous on $(0, \infty)$.

They denote by \mathcal{F} the family of all functions $F:\mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfy conditions (F_1) , (F_2) , and (F_3') . Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

F. Theorem 1.6[2]

Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ converges to x^* .

G. Theorem 1.7[2]

Let T be a self-mapping of a complete metric space X into itself. Suppose there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, [\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^* .

The aim of this paper is to introduce the modified generalized F -contractions, by combining the ideas of Dung and Hang [1], Piri and Kumam [2], Wardowski [3] and Wardowski and Van Dung [4] and give some fixed point result for these type mappings on complete metric space.

II. MAIN RESULTS

Let \mathcal{F}_G denote the family of all functions $F:\mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfy conditions (F_1) and (F_3') and \mathcal{F}_G denote the family of all functions $F:\mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfy conditions (F_1) and (F_3) .

A. Definition 2.1

Let (X, d) be a metric space and $T:X \rightarrow X$ be a mapping. T is said to be modified generalized F -contraction of type (A) if there exist \mathcal{F}_G and $\tau > 0$ such that

$$1) \quad \forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M_T(x, y))],$$

where

$$M_T(x, y) = \max\{d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) + d(x, Tx), d(Tx, y) + d(y, Ty)\}.$$

B. Remark 2.2

Note that \mathcal{F}_w . Since, for $\beta \in (0, \infty)$, the function $F(\alpha) = \frac{-1}{\alpha + \beta}$ satisfies the conditions (F_1) and (F_3') but it does not satisfy (F_2) , we have \mathcal{F}_w .

C. Definition 2.3

Let (X, d) be a metric space and $T: X \rightarrow X$ be a mapping. T is said to be modified generalized F -contraction of type (B) if there exist \mathcal{F}_G and $\tau > 0$ such that $\forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(M_T(x, y))]$.

D. Remark 2.4

Note that \mathcal{F}_w . Since, for $\beta \in (0, \infty)$, the function $F(\alpha) = \ln(\alpha + \beta)$ satisfies the conditions (F_1) and (F_3) but it does not satisfy (F_2) , we have $\mathcal{F} \subset \mathcal{F}_w$.

E. Remark 2.5

- 1) Every F -contraction is a modified generalized F -contraction.
- 2) Let T be a modified generalized F -contraction. From (3) for all $x, y \in X$ with $Tx \neq Ty$, we have

$$F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) \leq F(\max\{d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) + d(x, Tx), d(Tx, y) + d(y, Ty)\}).$$

Then, by (F_1) , we get

$$d(Tx, Ty) < \max\{d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) + d(x, Tx), d(Tx, y) + d(y, Ty)\}, \text{ for all } x, y \in X, Tx \neq Ty.$$

F. Theorem 2.6

Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a modified generalized F -contraction of type (A). Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n(x_0)\}_{n=1}^\infty$ converges to x^* .

G. Proof

Let $x_0 \in X$. Put $x_{n+1} = T^n x_0$ for all $n \in \mathbb{N}$. If, there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $Tx_n = x_n$. That is, x_n is a fixed point of T . Now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. It follows from (3) that, for all $n \in \mathbb{N}$,

$$\begin{aligned} & 1) \quad \tau + F(d(Tx_{n-1}, Tx_n)) \\ & \leq F(\max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}, \frac{d(T^2x_{n-1}, x_{n-1}) + d(T^2x_{n-1}, Tx_n)}{2}, d(T^2x_{n-1}, Tx_{n-1}), d(T^2x_{n-1}, x_n), d(T^2x_{n-1}, Tx_n) + d(x_{n-1}, Tx_{n-1}), d(Tx_{n-1}, x_n) + d(x_n, Tx_n)\}) \\ & = F(\max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}, \frac{d(x_{n+1}, x_{n-1}) + d(x_{n+1}, x_{n+1})}{2}, d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_n), d(x_{n+1}, x_{n+1}) + d(x_{n-1}, x_n), d(x_n, x_n) + d(x_n, x_{n+1})\}) \\ & = F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then (4) becomes

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})).$$

Since $\tau > 0$, we get a contradiction. Therefore

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n), \forall n \in \mathbb{N}.$$

Thus, from (4), we have

$$2) \quad F(d(x_n, x_{n+1})) \leq F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \tau F(d(x_{n-1}, x_n)).$$

It follows from (5) and (F_1) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \forall n \in \mathbb{N}.$$

Therefore $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ is a nonnegative decreasing sequence of real numbers, and hence

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \gamma \geq 0.$$

Now, we claim that $\gamma = 0$. Arguing by contradiction, we assume that $\gamma > 0$. Since $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ is a nonnegative decreasing sequence, for every $n \in \mathbb{N}$, we have

$$3) \quad d(x_{n+1}, x_n) \geq \gamma.$$

From (6) and (F_1) , we get

$$4) \quad F(\gamma) \leq F(d(x_{n+1}, x_n)) \leq F(d(x_{n-1}, x_n)) - \tau \leq F(d(x_{n-2}, x_{n-1})) - 2\tau$$

$$F(d(x_0, x_1)) - n\tau,$$

for all $n \in \mathbb{N}$. Since $F(\gamma) \in \mathbb{R}$ and $\lim_{n \rightarrow \infty} [F(d(x_0, x_1)) - n\tau] = -\infty$, there exists $n_1 \in \mathbb{N}$ such that

$$5) \quad F(d(x_0, x_1)) - n\tau < F(\gamma), \forall n > n_1.$$

It follows from (7) and (8) that

$$F(\gamma) \leq F(d(x_0, x_1)) - n\tau < F(\gamma), \forall n > n_1.$$

It is a contradiction. Therefore, we have

$$6) \quad \lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

As in the proof of Theorem 2.8 in [2], we can prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. So by completeness of (X, d) , $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X . Therefore,

7) $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$.

$$7) \quad \lim_{n \rightarrow \infty} d(x_n, x^*) = 0.$$

Finally, we will show that $x^* = Tx^*$. We only have the following two cases:

$$a) \quad \forall n \in \mathbb{N}, \exists i_n \in \mathbb{N}, i_n > i_{n-1}, i_0 = 1 \text{ and } x_{i_{n+1}} = Tx^*,$$

$$\exists n_3 \in \mathbb{N}, \forall n \geq n_3, d(Tx_n, Tx^*) > 0.$$

In the first case, we have

$$x^* = \lim_{n \rightarrow \infty} x_{i_{n+1}} = \lim_{n \rightarrow \infty} Tx^* = Tx^*.$$

In the second case from the assumption of Theorem 2.8, for all $n \geq n_3$, we have

$$8) \quad \tau + F(d(x_{n+1}, Tx^*)) = \tau + F(d(Tx_n, Tx^*))$$

$$\leq F(\max\{d(x_n, x^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}, \frac{d(T^2x_n, x_n) + d(T^2x_n, Tx^*)}{2}, d(T^2x_n, Tx_n), d(T^2x_n, x^*), d(T^2x_n, Tx^*) + d(x_n, Tx_n), d(Tx_n, x^*) + d(x^*, Tx^*)\}).$$

From (F_3') , (10), and taking the limit as $n \rightarrow \infty$ in (11), we obtain

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)).$$

This is a contradiction. Hence, $x^* = Tx^*$. Now, let us to show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ are two distinct fixed points of T , that is, $Tx^* = x^* \neq y^* = Ty^*$, then

$$d(Tx^*, Ty^*) = d(x^*, y^*) > 0.$$

It follows from (3) that

$$F(d(x^*, y^*)) < \tau + F(d(x^*, y^*))$$

$$= \tau + F(d(Tx^*, Ty^*))$$

$$\leq F(\max\{d(x^*, y^*), \frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{2}, \frac{d(T^2x^*, x^*) + d(T^2x^*, Ty^*)}{2}, d(T^2x^*, Tx^*), d(T^2x^*, y^*), d(T^2x^*, Ty^*) + d(x^*, Tx^*), d(Tx^*, y^*) + d(y^*, Ty^*)\}).$$

$$=F(\max\{d(x^*, y^*), \frac{d(x^*, y^*) + d(y^*, x^*)}{2}, \frac{d(x^*, x^*) + d(x^*, y^*)}{2}, d(x^*, x^*), d(x^*, y^*), d(x^*, y^*) + d(x^*, x^*), d(x^*, y^*) + d(y^*, x^*)\})$$

$$=F(d(x^*, y^*))$$

which is a contradiction. Therefore, the fixed point is unique.

REFERENCES

- [1] Dung, NV, Hang, VL: A fixed point theorem for generalized F-contractions on complete metric spaces. Vietnam J. Math. **43**, 743-753 (2015)
- [2] Piri, H, Kumam, P: Some fixed point theorems concerning F-contraction in complete metric spaces. Fixed Point Theory Appl. **2014**, 210 (2014). Doi
- [3] Wardowski, D: Fixed point theory of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
- [4] Wardowski, D, Van Dung, N: Fixed points of F-weak contractions on complete metric spaces. Demonstr. Math. **1**, 146-155 (2014).



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