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Fixed Point Result Satisfying Rational Contractive Condition in Metric Space

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Abstract: In this paper we prove some common fixed point results on complete metric space.
Keywords: complete metric space,$T-orbitally$ complete, weakly compatible, generalized weakly contractive

I. INTRODUCTION

The study of fixed point theorems and common fixed point theorems satisfying contractive type conditions has been a very active field of research activity during the last three decades. In 1922, the polish mathematician, Banach [14] proved a theorem which ensures under approximation conditions the existence and uniqueness of the fixe point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Many researchers have extended, generalized and improved Banach's fixed point theorem in different ways. Banach published first contractive definition for the fixed point theorem by using the concept of Lipschitz mapping which is known as Banach's contraction Principle. Final conclusion of the theorem is that T has a unique fixed point, which can be reached from any starting value $x_0 \in X$. Jungck [30] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps [31].

In 1997 Alber and Guerre- Delabriere [8] introduced the concept of weakly contractive map in Hilbert space and proved the existence of fixed point results. Rhoades [63] extended this concept in Banach space and established the existence of fixed points. Throughout this chapter (X, d) is a metric space which we denote simply by X.

II. PRELIMINARIES

A. Definition 1.1

For any $x_0 \in X$; $O(x_0) = \{T^n x_o ; n = 0,1,2,3 \dots\dots\}$ is said to the orbit of x_0 where,
$T^0 = I$, is the identity map of X. $\overline{O(x_0)}$ represent the closer of $O(x_0)$.

A metric space X is said to be $T-$orbitally complete; if every Cauchy sequence Which is contained in $O(x)$ for all $x \in X$ converges to the point of X.

Here we note that every complete metric space is $T-$ orbitally complete for any T, but converges is not true.

B. Definition 1.2

Let A and S be the mapping from a metric space X into itself, then the mapping is said to weakly compatible if they are commute at their coincidence points, that is,

$$Ax = Sx \text{ implies that } ASx = SAx.$$

C. Definition 1.3

A self map $T: X \to X$ is said to be generalized weakly contractive map if there exists a $\psi \in \Phi$ such that,

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y))$$

with $\lim_{t\to\infty} \psi(t) = 0$ for all $x, y \in X$.

We denote, $R^+ = [0, \infty)$ is positive real number, N the set of natural number and R the set of real number. We write $\Phi = \{\psi : R^+ \to R^+\}$ where ψ setisfies following conditions ;

1) ψ is continuous
2) ψ is non decreasing
3) $\psi(t) > 0$ for $t > 0$
4) $\psi(0) = 0$

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III. MAIN RESULT

A. Theorem 2.1

Let (X, d) be a $T-$ orbitally complete metric space, if A, B, S, T be the self mapping of X into itself such that;

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

2) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive map

3) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \leq k \max\left\{\frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx, Ty)\right\}.$$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof We suppose that, $x_0 \in X$ arbitrary and we choose a point $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1 \text{ and } y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \ldots\ldots\ldots$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 2.1(iii) we have,

$$d(y_{2n}, y_{2n+1}) \leq k.M(Ax_{2n}, Bx_{2n+1}) - \psi(M(Ax_{2n}, Bx_{2n+1}))$$

$$d(y_{2n}, y_{2n+1}) \leq k \max\left\{\begin{array}{c}\frac{d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})},\\ \frac{d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})},\\ d(Sx_{2n}, Tx_{2n+1})\end{array}\right\}$$

$$d(y_{2n}, y_{2n+1}) \leq k \max\left\{\begin{array}{c}\frac{d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ \frac{d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ d(y_{2n-1}, y_{2n})\end{array}\right\}$$

$$d(y_{2n}, y_{2n+1}) \leq k \max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\}$$

There arise three cases:

a) *Case- 1:* If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n})$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

b) *Case- 2:* If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n+1}, y_{2n})$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n})$$

which contradiction.

c) *Case- 3:* If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = 0$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq 0$$

which contradiction.

From the above all three cases we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

Processing the same way we have

$$d(y_{2n}, y_{2n+1}) \leq k^{2n}.d(y_0, y_1)$$

Or $\quad d(y_n, y_{n+1}) \leq k^n . d(y_0, y_1)$

For any $m > n$ we have

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots \ldots + d(y_{m-1}, y_m)$$
$$d(y_n, y_m) \leq (k^n + k^{n+1} + \ldots\ldots + k^{m-1}) d(y_0, y_1)$$
$$d(y_n, y_m) \leq \frac{k}{1-k} d(y_0, y_1) \ .$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1}$$
$$= \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y \ .$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 2.1(iii),

$$d(Ax_{2n}, Bv) \leq k \max \left\{ \begin{array}{c} \frac{d(Ax_{2n}, Sx_{2n})\, d(Bv, Tv)}{1 + d(Sx_{2n}, Tv)}, \\ \frac{d(Sx_{2n}, Bv) d(Ax_{2n}, Tv)}{1 + d(Sx_{2n}, Tv)}, \\ d(Sx_{2n}, Tv) \end{array} \right\}$$

$$d(y, Bv) \leq k \max\{ d(Bv, y), d(y, Bv), 0\}$$
$$d(y, Bv) < k \, . d(y, Bv)$$

which contradiction,

Hence $Bv = y = Tv$ and that $BTv = TBv$ implies that $By = Ty$.

Now we proof that $By = y$ for this again from 2.1(iii)

$$d(Ax_{2n}, By) \leq k \max \left\{ \begin{array}{c} \frac{d(Sx_{2n}, By) . d(Ax_{2n}, Ty)}{1 + d(Sx_{2n}, Ty)}, \\ \frac{d(Ax_{2n}, Sx_{2n}) . d(By, Ty)}{1 + d(Sx_{2n}, Ty)}, \\ d(Sx_{2n}, Ty) \end{array} \right\}$$

$$\lim_{n\to\infty} d(Ax_{2n}, By) \leq k\, d(y, By)$$
$$By = y = Ty \ .$$

Since $B(X) \subseteq S(X)$

for, $w \in X$ such that $Sw = y$

now we show that $Aw = y$

$$d(Aw, By) \leq k \max\left\{ \frac{d(Aw, Sw) d(By, Ty)}{1 + d(Sw, Ty)}, \frac{d(Sw, By) d(Aw, Ty)}{1 + d(Sw, Ty)}, d(Sw, Ty) \right\}$$

It follows that, $\quad d(Aw, y) \leq kd(Aw, y)$

Which contradiction, $d(Aw, y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ for this again from 2.1(iii),

$$d(Aw, By) \leq k \max \left\{ \frac{d(Ay, Sy)\, d(By, Ty)}{1 + d(Sy, Ty)}, \frac{d(Sy, By) d(Ay, Ty)}{1 + d(Sy, Ty)}, d(Sy, Ty) \right\}$$

It follows that, $d(Ay, y) \leq k\, d(Ay, y)$

Which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

that is y is common fixed point of A, B, S, T.

If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness We suppose that x, is another fixed point for A, B, S, T then, by using 2.1(iii) then we have

$$d(x, y) \leq k . d(x, y)$$

Which contradiction. so that $x = y$ and y is unique fixed point of A, B, S, T.

This complete the prove of the theorem.

If we omit the completeness of the space then we get following corollary.

B. Corollary 2.2

Let (X, d) be a $T-$ orbitally metric space, if A, B, S, T be the self mapping of X into itself such that;

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.
2) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive ma
3) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \leq k.M(x, y) - \psi(M(x, y))$$

Where , $M(Ax, By) = \max\left\{\begin{array}{l} \frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Sx,Ty)}, \frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,Ty)}, \\ \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx,Ty) \end{array}\right\}.$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

C. *Corollary 2.3*

Let (X, d) be a $T-$ *orbitally* complete metric space, if A, Bbe the self mapping of X into itself such that;

1) $A(X) \subseteq X$ and $B(X) \subseteq X$,
2) The pair (A, B) weakly compatible and generalized weakly contractive map
3) *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax, By) \leq k.M(x, y) - \psi(M(x, y))$$

Where , $M(Ax, By) = max\left\{\begin{array}{l} \frac{d^2(Ax,x)+d^2(By,y)}{1+d(x,y)}, \frac{d^2(x,By)+d^2(Ax,y)}{1+d(x,y)}, \\ \frac{d(Ax,x).d(By,y)}{1+d(x,y)}, \frac{d(x,By).d(Ax,y)}{1+d(x,y)}, d(x,y) \end{array}\right\}.$

Then A, B have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is enough if we take $S = T = I$ (identity mapping) in Theorem 2.1 then we get the result.

D. *Corollary 2.4*

Let (X, d) be a $T-$ *orbitally* complete metric space, if A, Bbe the self mapping of X into itself such that *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax, Ay) \leq k.M(x, y) - \psi(M(x, y))$$

Where , $M(Ax, Ay) = \max\left\{\begin{array}{l} \frac{d^2(Ax,x)+d^2(Ay,y)}{1+d(x,y)}, \frac{d^2(x,Ay)+d^2(Ax,y)}{1+d(x,y)}, \\ \frac{d(Ax,x).d(Ay,y)}{1+d(x,y)}, \frac{d(x,Ay).d(Ax,y)}{1+d(x,y)}, d(x,y) \end{array}\right\}.$

Then A, B have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is enough if we take $A = B$ in Corollary 2.3 then we get the result.

E. *Corollary 2.5*

Let (X, d) be a $T-$ *orbitally* complete metric space, if A, B, S, T be the self mapping of X into itself such that;

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X
2) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive ma
3) *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax, By) \leq k.max\left\{\begin{array}{l} \frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Sx,Ty)}, \frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,Ty)}, \\ \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx,Ty) \end{array}\right\}$$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is immediate to see that if we take $\psi(t) = 0$ in Theorem 2.1, then we get the result.

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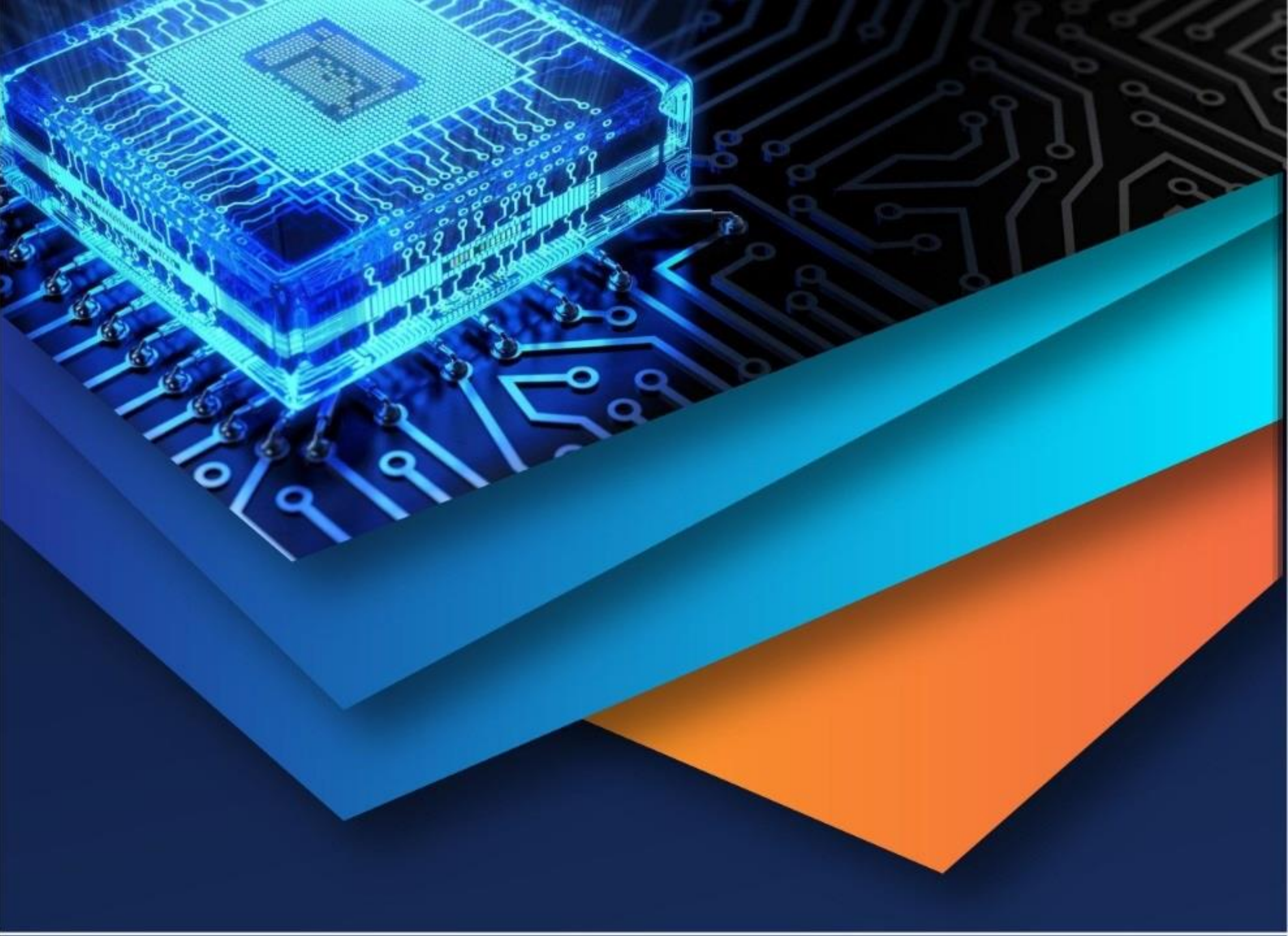

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