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Integral Solutions of the Homogeneous Biquadratic Diophantine Equation

$$3(x^4 - y^4) - 2xy(x^2 - y^2) = 972(z + w)p^3$$

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Abstract: ***The Homogeneous Biquadratic Diophantine equation with five unknowns represented by*** $3(x^4 - y^4) - 2xy(x^2 - y^2) = 972\,(z + w)p^3$ ***is analyzed for its non-zero distinct integer solutions. Different patterns of integral solutions satisfying the equation are obtained. A few interesting relations between the solutions and some special numbers are presented.***
Keywords: Biquadratic equation with five unknowns, Integral solutions

I. INTRODUCTION

Mathematics is the language of patterns and relationships and is used to describe anything that can be quantified. Diophantine equations have stimulated the interest of various mathematicians. Diophantine equations with higher degree greater than three can be reduced in to equations of degree 2 or 3 and it can be easily solved. In [1-3], theory of numbers were discussed. In [4-5], quadratic diophantine equations are discussed. In [6-10], cubic, biquadratic and higher order equations are considered for its integral solutions.

In this communication a homogeneous Biquadratic Diophantine equation, with five variables represented by $3(x^4 - y^4) - 2xy(x^2 - y^2) = 972\ (z + w)p^3$ is considered and in particular a few interesting relations among the solutions are presented.

A. Notations

Obl_n = Oblong number of rank ‘n’.

P_n^m = Pyramidal number of rank ‘n’ with sides ‘m’.

$T_{m,n}$ = Polygonal number of rank ‘n’ with sides ‘m’.

CS_n = Centered Square number of rank ‘n’.

SO_n = Stella octangula number of rank ‘n’.

O_n = Octahedral number of rank ‘n’.

Gno_n = Gnomonic number of rank ‘n’.

$Star_n$ = Star number of rank ‘n’.

Tha_n = Thabit-ibn-kurrah number of rank ‘n’.

$4DF_n$ = Four Dimensional Figurate number whose generating Polygon is a square.

$Carl_n$ = Carol number of rank ‘n’.

Nex_n = Nexus number of rank ‘n’.

K_n = Kynea number of rank ‘n’.

j_n = Jacobsthal-Lucas number of rank ‘n’.

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TO_n = Truncated Octahedral number of rank 'n'.

TT_n = Truncated Tetrahedral number of rank 'n'.

CH_n = Centered Hexagonal number of rank 'n'.

J_n = Jacobsthal number of rank 'n'.

B. *Method of Analysis*

The Biquadratic Diophantine equation to be solved for its non-zero integral solution is

$$3(x^4 - y^4) - 2xy(x^2 - y^2) = 972\ (z + w)p^3 \qquad (1)$$

On substitution of the transformations,

$$x = u + v\ ,\ y = u - v,\ z = 2uv + 1 \text{ and } w = 2uv - 1 \qquad (2)$$

in (1) leads to, $u^2 + 2v^2 = 243\ p^3$ (3)

Four different patterns of non-zero distinct integer solutions to (1) are illustrated below:

1) *Pattern: 1:*

Assume $p = p(a,b) = a^2 + 2b^2$ (4)

where a and b are non-zero integers.

and write $243 = (15 + 3i\sqrt{2})(15 - 3i\sqrt{2})$ (5)

Substituting (4) & (5) in (3), and using factorization method,

$$(u + i\sqrt{2}v)(u - i\sqrt{2}v) = (15 + 3i\sqrt{2})(15 - 3i\sqrt{2})\left((a + i\sqrt{2}b)^3 (a - i\sqrt{2}b)^3\right) \qquad (6)$$

Equating the like terms and comparing real and imaginary parts, we get

$$u = u(a,b) = 15a^3 - 18a^2b - 90\ ab^2 + 24\ b^3$$
$$v = v(a,b) = 3a^3 + 45a^2b - 18\ ab^2 - 60\ b^3$$

Substituting the above values of u & v in equation (2), the corresponding integer solutions of (1) are given by

$$\left.\begin{aligned}
x &= x(a,b) = 18a^3 + 27a^2b - 108\ ab^2 - 36\ b^3\\
y &= y(a,b) = 12a^3 - 63a^2b - 72\ ab^2 + 84\ b^3\\
z &= z(a,b) = 2\left(45\ a^6 + 621a^5b - 1350\ a^4b^2 - 4554\ a^3b^3 + 3780\ a^2b^4 + 4968\ ab^5 - 1440\ b^6\right) + 1\\
w &= w(a,b) = 2\left(45\ a^6 + 621a^5b - 1350\ a^4b^2 - 4554\ a^3b^3 + 3780\ a^2b^4 + 4968\ ab^5 - 1440\ b^6\right) - 1\\
p &= p(a,b) = a^2 + 2b^2
\end{aligned}\right\} \quad (7)$$

Properties

a) $x(a,a) - y(a,a) + p(a,a) + z(a,a) - w(a,a) + 360\ P_a^3 - T_{8,a} + 29\ Gno_a \equiv 0 \pmod{27}$

b) $p(a,a)[z(a,a) - w(a,a)]$ is a nasty number.

c) $z(1,1) + w(1,1) - p(1,1)\,[y(1,1) - x(1,1)]$ is a perfect square.

d) $x(b,b) - y(b,b) - p(b,b) + 90\ O_b + T_{8,b} \equiv 0 \pmod{28}$

e) $3y(2^n,1) - 2x(2^n,1) + 243\ K_n - 162\ Tha_n \equiv 0 \pmod{243}$

2) *Pattern: 2:*

Instead of (5), write $243 = (1 + 11\,i\sqrt{2})(1 - 11\,i\sqrt{2})$ (8)

Substituting (4) and (8) in (3), and employing the method of factorization, following the procedure presented in pattern 1, the corresponding integer solutions of (1) are represented by

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$$\left.\begin{aligned}
x &= x(a,b) = 12a^3 - 63a^2b - 72ab^2 + 84b^3\\
y &= y(a,b) = -10a^3 - 69a^2b + 60ab^2 + 92b^3\\
z &= z(a,b) = 2\left(11a^6 - 723a^5b - 66a^4b^2 + 5302a^3b^3 + 660a^2b^4 - 5784ab^5 - 352b^6\right) + 1\\
w &= w(a,b) = 2\left(11a^6 - 723a^5b - 66a^4b^2 + 5302a^3b^3 + 660a^2b^4 - 5784ab^5 - 352b^6\right) - 1\\
p &= p(a,b) = a^2 + 2b^2
\end{aligned}\right\} \quad (9)$$

Properties

a) $a\,x(a,1) - 2Nex_a - 24(4DF_a) + 166P_a^5 + T_{16,a} - 34Gno_a \equiv 0 \pmod{32}$

b) $z(a,1) - w(a,1) + p(a,1) - 2T_{6,a} + T_{12,a} - CS_a \equiv 0 \pmod{3}$

c) $y(a,1) - x(a,1) - p(a,1) + TO_a + 3SO_a + 2(T_{14,a} + T_{30,a}) \equiv 0 \pmod{117}$

d) $5x(2^n,1) + 6y(2^n,1) + p(2^n,1) - 728(j_n + 3J_n - K_n) \equiv 0 \pmod{246}$

e) $z(a,a) - w(a,a) - p(a,a) + Star_a - T_{12,a} + T_{8,a} - CH_a + 2\,Obl_a + 2Gno_a \equiv 0 \pmod{5}$

3) Pattern: 3:

Writing $243 = (9 + 9i\sqrt{2})(9 - 9i\sqrt{2})$ (10)

The corresponding integer solutions of (1) are represented by

$$\left.\begin{aligned}
x &= x(a,b) = 18a^3 - 27a^2b - 108ab^2 + 36b^3\\
y &= y(a,b) = -81a^2b + 108b^3\\
z &= z(a,b) = 2\left(81a^6 - 243a^5b - 2430a^4b^2 + 1782a^3b^3 + 6804a^2b^4 - 1944ab^5 - 2592b^6\right) + 1\\
w &= w(a,b) = 2\left(81a^6 - 243a^5b - 2430a^4b^2 + 1782a^3b^3 + 6804a^2b^4 - 1944ab^5 - 2592b^6\right) - 1\\
p &= p(a,b) = a^2 + 2b^2
\end{aligned}\right\} \quad (11)$$

Properties

a) $y(a,a) - x(a,a) - p(a,a) - 216P_a^5 + 18Star_a + T_{8,a} + 55Gno_a \equiv 0 \pmod{37}$

b) $z(a,1) - w(a,1) - p(a,1) - CH_a + CS_a + 2Obl_a \equiv 0 \pmod{3}$

c) $p(a,1) + CS_a - 6TT_a + 2T_{22,a} - 7Gno_a \equiv 0 \pmod{7}$

d) $y(2^n,1) + p(2^n,1) - 80(j_n + 3J_n - K_n) \equiv 0 \pmod{30}$

e) $z(2^n,1) - w(2^n,1) - 3p(2^n,1) + 2Tha_n + 3Carl_n \equiv 0 \pmod{9}$

4) Pattern: 4:

Instead of (5), write $243 = \dfrac{(43 + 13i\sqrt{2})(43 - 13i\sqrt{2})}{9}$ (12)

Substituting (12) and (4) in (3) and employing the method of factorization, following the procedure presented in pattern 1, the corresponding integer solutions of (3) are represented by

$$u = u(a,b) = \frac{1}{3}\left(43a^3 - 78a^2b - 258ab^2 + 104b^3\right)$$

$$v = v(a,b) = \frac{1}{3}\left(13a^3 + 129a^2b - 78ab^2 - 172b^3\right)$$

Since our interest is on finding integer solutions, we have choose a and b suitably so that u and v are integers. Let us take $a = 3A$ and $b = 3B$,

$u = u(A,B) = 387A^3 - 702A^2B - 2322AB^2 + 936B^3$

$v = v(A,B) = 117A^3 - 1161A^2B - 702AB^2 - 1548B^3$

In view of (2), the integer solutions of (1) are given by

$$\left.\begin{array}{l}
x = x(A,B) = 504A^3 + 459A^2B - 3024AB^2 - 612B^3 \\
y = y(A,B) = 270A^3 - 1863A^2B - 1620AB^2 + 2484B^3 \\
z = z(A,B) = 2\begin{pmatrix} 45279A^6 + 367173A^5B - 1358370A^4B^2 - 2692602A^3B^3 \\ + 3803436A^2B^4 + 2937384AB^5 - 1448928B^6 \end{pmatrix} + 1 \\
w = w(A,B) = 2\begin{pmatrix} 45279A^6 + 367173A^5B - 1358370A^4B^2 - 2692602A^3B^3 \\ + 3803436A^2B^4 + 2937384AB^5 - 1448928B^6 \end{pmatrix} - 1 \\
p = p(A,B) = 9A^2 + 18B^2
\end{array}\right\} \quad (13)$$

Properties

a) $x(A,A) + y(A,A) + p(A,A) + 135(15O_A + 5SO_A + 10P_A^5 - T_{12,A}) - 270Gno_A \equiv 0 \pmod{270}$

b) $w(A,A) - z(A,A) + p(A,A) - 6TT_A - Star_A + CS_A - 12Gno_A \equiv 0 \pmod{7}$

c) $CS_A + 5T_{12,A} - p(A,A) + 11Gno_A \equiv 0 \pmod{10}$

d) $z(2^n,1) - w(2^n,1) + p(2^n,1) + 9(j_n + 3J_n - K_n) \equiv 0 \pmod{29}$

e) $z(A,1) - w(A,1) + p(A,1) + Obl_A - Star_A - 2CS_A + 17Gno_A \equiv 0 \pmod{45}$

Note:

In addition, one may write 243 as

$$243 = \left\{\begin{array}{l}
\dfrac{(125 + 83i\sqrt{2})(125 - 83i\sqrt{2})}{121} \\
\dfrac{(141 + 69i\sqrt{2})(141 - 69i\sqrt{2})}{121} \\
\dfrac{(139 + 71i\sqrt{2})(139 - 71i\sqrt{2})}{121} \\
\dfrac{(69 + 111i\sqrt{2})(69 - 111i\sqrt{2})}{121} \\
\dfrac{(171 + 9i\sqrt{2})(171 - 9i\sqrt{2})}{121} \\
\dfrac{(45 + 117i\sqrt{2})(45 - 117i\sqrt{2})}{121}
\end{array}\right.$$

For these choices, one may obtain different patterns of solutions of (1).

II. CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous Biquadratic Diophantine equation with five unknowns. As the equations are rich in variety, one may search for other forms of biquadratic equations with many variables and obtain their corresponding properties.

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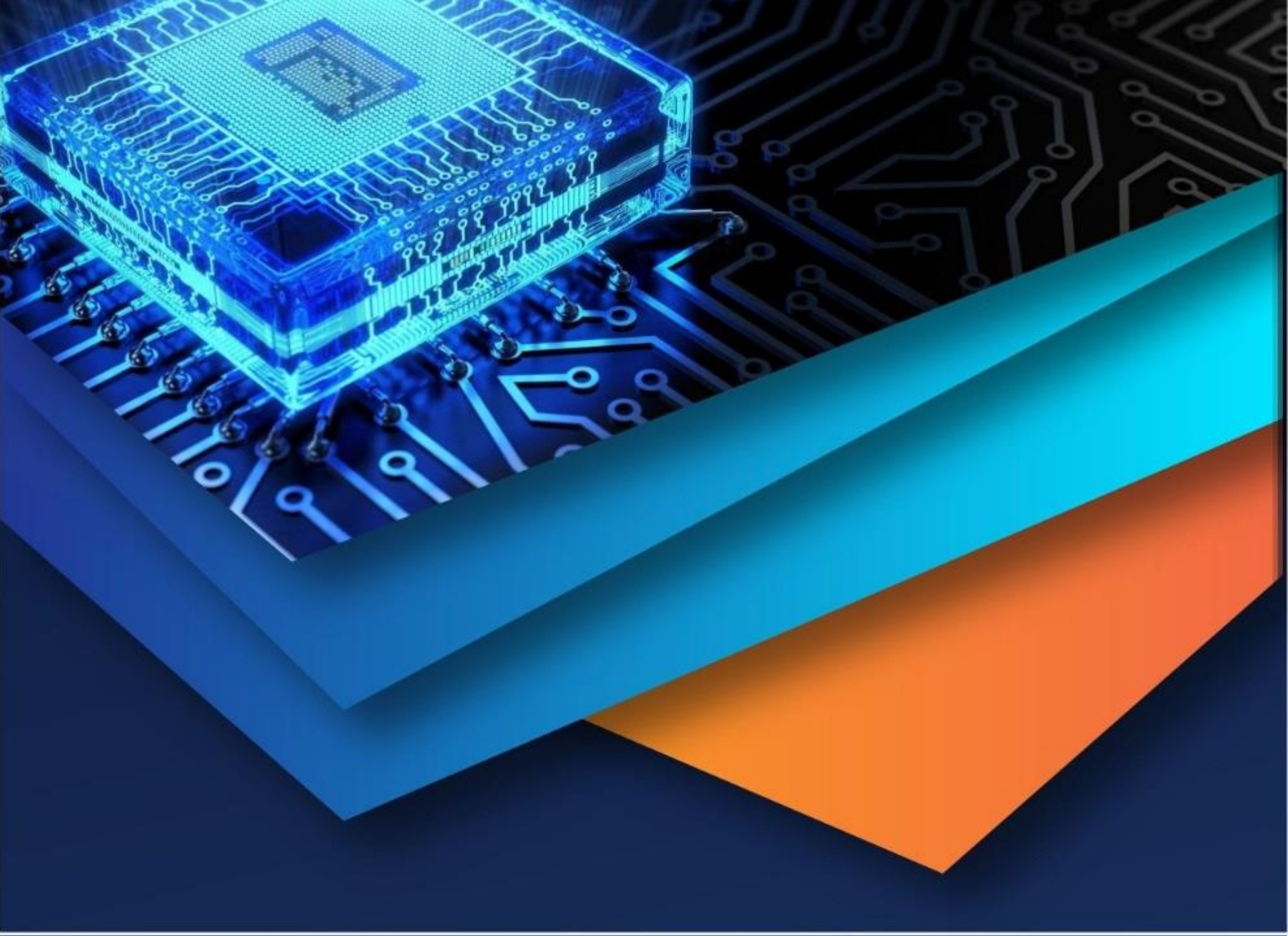

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